TOTALLY RAMIFIED VALUATIONS
ON FINITE-DIMENSIONAL DIVISION ALGEBRAS

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ABSTRACT. Division algebras D with valuation v are studied, where D is finite-dimensional and totally ramified over its center F (i.e., the ramification index of v over $v|_F$ equals $[D:F]$). Such division algebras have appeared in some important constructions, but the structure of these algebras has not been systematically analyzed before. When $v|_F$ is Henselian a full classification of the F-subalgebras of D is given. When F has a Henselian valuation v with separably closed residue field and A is any tame central simple F-algebra, an algorithm is given for computing the underlying division algebra of A from a suitable subgroup of A^*/F^*. Some examples are constructed using this valuation theory, including the first example of finite-dimensional F-central division algebras D_1 and D_2 with $D_1 \otimes_F D_2$ not a division ring, but D_1 and D_2 having no common subfield $K \supsetneq F$.

Valuation theory, long a basic tool in commutative algebra, has been relatively neglected in the study of division algebras, until quite recently. Nonetheless, valuations are naturally present in a number of division algebras that have been constructed to exhibit special properties, particularly algebras over iterated Laurent power series fields. For example, such division algebras have been key ingredients in Amitsur's noncrossed product construction [**Am**] and in Platonov's construction [**P**] of division algebras D with $SK_1(D) \neq 1$. Valuations are not so prevalent on division algebras as on fields. But if a division algebra D does have a valuation, this structure contains a substantial amount of information about D which would scarcely be accessible otherwise.

We consider here valued division algebras D for which D is *totally ramified* and *tame* over its center F, i.e., for which the ramification index $|\Gamma_D : \Gamma_F|$ equals the dimension $[D:F]$ of D over F and the characteristic $\mathrm{char}(\overline{D})$ does not divide $[D:F]$. (Here Γ_D is the value group of the valuation on D, and $\overline{D}$ is the residue division algebra. We assume throughout that $[D:F] < \infty$.) Valued division algebras of this type appear, e.g., in Amitsur's noncrossed product paper [**Am**, §2], in Saltman's work on indecomposable division algebras [**Sa**], in certain of the Malcev-Neumann division algebras considered by the first author and Amitsur [**TA₂**,§4], etc. However, the intrinsic structure of totally ramified tame division algebras has apparently not been examined closely before. This may be because most past work on valued division algebras has concentrated on discrete valuations, when $\Gamma_D \cong \mathbf{Z}$; for such a valuation D is never totally ramified over its center F unless $D = F$ (cf. (3.2) below).

If D is tame and totally ramified over its center F with respect to a valuation v, we will prove that there is a natural nondegenerate symplectic pairing on the relative value group Γ_D/Γ_F. We will show how this pairing limits the possible decompositions of D into tensor products of cyclic algebras (Proposition 3.3) and will show (Theorem 3.8) that if v is Henselian on F, then the subgroups of Γ_D/Γ_F completely classify the isomorphism classes of F-subalgebras of D. If (F,v) is strictly Henselian (i.e., Henselian with $\overline{F}$ separably closed), we provide in §4 an algorithm for determining the underlying division algebra of a central simple F-algebra. Among other things, this allows one to calculate the underlying division algebra of a tensor product of two F-central division algebras (Example 4.4(i)). Every tame division algebra D over such an F is necessarily totally ramified, and our algorithm allows a full determination of which fields algebraic over F split D (Proposition 4.5).

The main results on totally ramified tame division algebras are given in §§3–4, with preliminary machinery and terminology set up in §§1–2. In §5 we apply the general theory to a couple of examples. Notably, we give an example of division algebras D_1 and D_2 with center F such that $D_1 \otimes_F D_2$ is not a division ring but there is no proper extension field of F lying in both D_1 and D_2. This settles an old question in the theory of algebras.

1. Valued division algebras. In this section we establish basic terminology and recall some of the essential facts about valuations on division rings and fields that will be used frequently. We also prove a convenient extension lemma for a valued division algebra totally ramified over its center.

Every ring A considered here is assumed to be a finite-dimensional algebra over some field. ("Field" means commutative field.) We assume always that A has a 1, and write A^* for the group of units of A. The center of A is denoted $Z(A)$. We write $|S|$ for the cardinality of a set S.

Let D be a division ring. A *valuation* on D is a function $v: D^* \to G$, where G is a totally ordered abelian group (written additively), such that for all $a, b \in D^*$,

(i) $v(ab) = v(a) + v(b)$;

(ii) $v(a + b) \geq \min(v(a), v(b))$ if $b \neq -a$.

The standard reference for valuation theory on division rings is Schilling's book [**S**]. (Schilling does not require the ordered group G to be abelian. However, since we are assuming that D is a finite-dimensional algebra over a field an easy argument (cf. [**W₂**]) shows that $v(D^*)$ is abelian even if G is not abelian.) Let $\Gamma_D := v(D^*)$, the value group of v on D; $V_D := \{a \in D^* \mid v(a) \geq 0\} \cup \{0\}$, the valuation ring of v; $M_D := \{a \in D^* \mid v(a) > 0\} \cup \{0\}$, the unique maximal left ideal and maximal right ideal of V_D; $U_D := V_D - M_D = V_D^*$, the group of v-units; and $\overline{D} := V_D/M_D$, the residue division ring of v. When (not until §5) we consider more than one valuation on the same division ring, we will write $\Gamma_{D,v}, \ldots, V_{D,v}, \overline{D}_v$. For $a \in V_D$ we write $\bar{a}$ for the image of a in $\overline{D}$.

Let E be any division ring which is a subring of the valued division ring (D, v). The restriction $v|_E$ of v to E^* is clearly a valuation of E. Then Γ_E is a subgroup of Γ_D and $|\Gamma_D : \Gamma_E|$ is the *ramification index* of D over E with respect to v. Also $V_D \cap E = V_E$ and $M_D \cap E = M_E$, so we can view $\overline{E}$ as a subring of $\overline{D}$. The *residue degree* of D over E is $[\overline{D} : \overline{E}]$, which is the dimension of $\overline{D}$ as a left $\overline{E}$-vector space.

Recall (e.g from [**S**, pp. 21–22]) the "fundamental inequality"

$$(1.1) \qquad |\Gamma_D : \Gamma_E| \cdot [\overline{D} : \overline{E}] \leq [D : E].$$

D is said to be "defectless" over E if $[D : E] < \infty$ and equality holds in (1.1). (For example, it can be shown that if $\Gamma_D \cong \mathbf{Z}$, then D is defectless over $Z(D)$. We omit the proof, since we will have no occasion to use this fact.) Assume $[D : E] < \infty$. We say that D is *totally ramified* over E with respect to v if $|\Gamma_D : \Gamma_E| = [D : E]$. When this occurs, (1.1) shows that $\overline{D} = \overline{E}$. At the other extreme, if $[\overline{D} : \overline{E}] = [D : E]$ we say that D is *inertial* over E. Finally, we say D is *tame* over E if char$(\overline{E}) \nmid [D : E]$. In his posthumous paper [**D₂**] P. Draxl proved a "noncommutative Ostrowski theorem," which is a noteworthy refinement of (1.1). Draxl's theorem [**D₂**, Theorem 2] says: If the restriction of v to $Z(E)$ is Henselian, then

$$(1.2) \qquad [D : E] = [\overline{D} : \overline{E}] \cdot |\Gamma_D : \Gamma_E| \cdot p^b,$$

where $p = \text{char}(\overline{E})$ ($p = 1$ if char$(\overline{E}) = 0$).

Given division rings $E \subseteq D$, suppose v is a valuation on E. We say that v *extends to D* if there is a valuation w on D with $w|_E = v$. When v has a unique extension to D (i.e., unique up to the choice of $\Gamma_D \supseteq \Gamma_E$) we will routinely use the same letter v for the extended valuation. Let Δ be the divisible hull of the torsion-free abelian group Γ_E. Then $\Delta \cong \Gamma_E \otimes_{\mathbf{Z}} \mathbf{Q}$, and the ordering on Γ_E extends uniquely to a total ordering on Δ. Suppose $[D : E] < \infty$ and v extends to a valuation on D; then as Γ_D / Γ_E is torsion and Γ_D is torsion-free there is a unique monomorphism $f : \Gamma_D \to \Delta$ which restricts to the injection $\Gamma_E \hookrightarrow \Delta$, and f is order-preserving. In light of this, we will view Γ_D as a subgroup of Δ, thereby removing the ambiguity about the choice of $\Gamma_D \supseteq \Gamma_E$.

Unlike the situation with fields, it is *not* always true that every valuation on the division algebra E can be extended to a valuation on $D \supseteq E$. This has been a serious obstacle in the development of noncommutative valuation theory, and has led to consideration of other valuation-like conditions with better extension properties (cf. [**VG**]). There is a general criterion given by M. Krasner and P. M. Cohn (cf. [**VG**, p. 31] or [**CM**, Theorem 2.3]) for when a valuation can be extended from E to D. Another criterion has been given for the special case $E = Z(D)$ (cf. [**W₂**; **C**, Theorem 1]):

VALUATION EXTENSION CRITERION 1.3. *Let D be a division ring (with $[D : Z(D)] < \infty$, as always) and let v be a valuation on $Z(D)$. Then v has an extension to a valuation on D if and only if v extends uniquely to each field K with $Z(D) \subseteq K \subseteq D$.*

It is immediate from this criterion that if v is any valuation on $Z(D)$, then v has at most one extension to D. Furthermore, the criterion indicates why most noncommutative valuation theory has been carried out on division algebras over Henselian fields. For, recall [**E**, (16.4), (16.6)] that a valuation v on a field F is Henselian if and only if v has a unique extension to each field K algebraic over F. Thus, if (F, v) is Henselian and a division ring D is a finite-dimensional F-algebra, then v has a unique extension to D. For, v extends uniquely to $Z(D)$, and $(Z(D), v)$ is also Henselian, so (1.3) applies.

In our study of totally ramified tame division algebras we will make heavy use of the basic facts about totally ramified tame field extensions. We recall those facts below, after introducing some more terminology.

For any field F and any positive integer l write $\mu(F)$ for the group of roots of unity of F and $\mu_l(F)$ for the cyclic group of lth roots of unity of F. We write $\mu_l \subseteq F$ in short for: F contains l lth roots of unity. (This can only occur if $\operatorname{char}(F) \nmid l$.) The notation $\mu_l(F)$ will only be used when $\mu_l \subseteq F$.

If G is a finite abelian group, let $\hat{G} := \operatorname{Hom}(G, \mathbf{Q}/\mathbf{Z})$, the dual group of G. Recall that $G \cong \hat{G}$, though not canonically. For $g \in G$, we write $\operatorname{ord}(g)$ for the order of g in G; let $\exp(G) := \operatorname{lcm}\{\operatorname{ord}(g) \mid g \in G\}$, the exponent of G. If G_1, G_2, G_3 are groups with identity elements 1_{G_i}, then a function $f\colon G_1 \times G_2 \to G_3$ is said to be *nondegenerate* if for each $g_i \in G_i$, $g_i \neq 1_{G_i}$, $i = 1, 2$, there is an $h_j \in G_j$, $j = 3 - i$, with $f(g_1, h_2) \neq 1_{G_3}$ and $f(h_1, g_2) \neq 1_{G_3}$.

Now, suppose $F \subseteq K$ are fields with $[K : F] < \infty$. Let v be a valuation on K with respect to which K is tame and totally ramified over F. Let $l = \exp(\Gamma_K/\Gamma_F)$. Then, recall

PROPOSITION 1.4. (i) *Suppose K is Galois over F, with $\mathcal{G}(K/F)$ the Galois group. Then $\mu_l \subseteq \overline{F}$, and there is a well-defined nondegenerate bilinear pairing*

$$\gamma\colon (\Gamma_K/\Gamma_F) \times \mathcal{G}(K/F) \to \mu_l(\overline{F}) \quad \text{given by} \quad \gamma(v(c) + \Gamma_F, \tau) = \overline{c/\tau(c)} \in \overline{F}$$

for all $c \in K^*$ and $\tau \in \mathcal{G}(K/F)$. The pairing induces an isomorphism $\mathcal{G}(K/F) \cong \widehat{\Gamma_K/\Gamma_F}$. Hence, $\mathcal{G}(K/F)$ is an abelian group with $\mathcal{G}(K/F) \cong \Gamma_K/\Gamma_F$.

(ii) *K is a Kummer extension of F if and only if K is Galois over F and $\mu_l \subseteq F$. When this occurs, let $S = \{a \in K^* \mid a^l \in F^*\}$, the Kummer set of K over F. Then the map $\bar{v}\colon S/F^* \to \Gamma_K/\Gamma_F$ induced by v is an isomorphism, and the diagram*

$$
(1.5) \qquad
\begin{array}{ccc}
(S/F^*) \times \mathcal{G}(K/F) & \longrightarrow & \mu_l(F) \\
{\scriptstyle(\bar{v},\operatorname{id})}\downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\
(\Gamma_K/\Gamma_F) \times \mathcal{G}(K/F) & \xrightarrow{\ \gamma\ } & \mu_l(\overline{F})
\end{array}
$$

is commutative, where the upper map is the nondegenerate Kummer pairing given by $(aF^*, \tau) \mapsto a/\tau(a)$.

(iii) *Suppose (F, v) is Henselian. Then K is a compositum of radical extensions of F (even if K/F is not Galois). Thus, K is a Kummer extension if and only if $\mu_l \subseteq F$ if and only if $\mu_l \subseteq \overline{F}$.*

For a proof of (1.4)(i), see [**E**, (20.11), pp. 161–162]. The first sentence of (ii) is immediate from (i). The injectivity of $\bar{v}$ follows from the obvious commutativity of diagram (1.5) and the nondegeneracy of the Kummer pairing (cf. [**J$_1$**, pp. 119–123]). Then $\bar{v}$ is surjective since $|\Gamma_K : \Gamma_F| = [K : F] = |S/F^*|$. For (iii), see [**S**, p. 64, Theorem 3].

Valued division algebras are often easier to work with when the valuation on the center is Henselian with sufficiently large residue field. The next lemma provides a device for reducing to that case when the division algebra is totally ramified over its center.

EXTENSION LEMMA 1.6. *Let F be a field with valuation v. Let D_1 and D_2 be division algebras over F with respective valuations w_1 and w_2 extending v on F.*

Suppose $[D_1 : F] < \infty$, D_1 is totally ramified over F, and $\Gamma_{D_1} \cap \Gamma_{D_2} = \Gamma_F$. Then $D_1 \otimes_F D_2$ is a division ring with a unique valuation w extending w_1 on $D_1 \otimes F$ and w_2 on $F \otimes D_2$. The value group of w is $\Gamma_{D_1} + \Gamma_{D_2}$ and $D_1 \otimes D_2$ is totally ramified over D_2.

PROOF. Let Δ be the divisible hull of Γ_{D_2}. Since Γ_{D_1}/Γ_F is torsion, the injection $\Gamma_F \hookrightarrow \Gamma_{D_2} \hookrightarrow \Delta$ extends uniquely to an injection $\Gamma_{D_1} \hookrightarrow \Delta$, which is order-preserving. In computing $\Gamma_{D_1} \cap \Gamma_{D_2}$ and $\Gamma_{D_1} + \Gamma_{D_2}$ we view Γ_{D_1} as a subgroup of Δ.

Choose and fix inverse images $b_1, \ldots, b_k$ in D_1^* of all the distinct elements of Γ_{D_1}/Γ_F. Then in any linear combination $\sum_{i=1}^{k} c_i b_i$ with $c_i \in F$, the nonzero summands $c_i b_i$ all have different values under w_1. Thus, the b_i are F-linearly independent, so (by dimension count) they form an F-basis of D_1. Also in any nonzero F-linear combination $\sum c_i b_i$, since the minimum value among the nonzero summands occurs just once, $w_1(\sum c_i b_i) = \inf\{w_1(c_i b_i)| \ c_i \neq 0\}$.

Every element $d \in D_1 \otimes_F D_2$ has a unique representation $d = \sum_{i=1}^{k} b_i \otimes r_i$, with each $r_i \in D_2$. If $D_1 \otimes D_2$ has a valuation w extending w_1 and w_2, then for $r_i \neq 0$, $w(b_i \otimes r_i) = w((b_i \otimes 1) \cdot (1 \otimes r_i)) = w_1(b_i) + w_2(r_i) \in \Delta$. Since Γ_{D_1}/Γ_F injects into Δ/Γ_{D_2}, $w(b_i \otimes r_i) \neq w(b_j \otimes r_j)$ for $i \neq j$. Thus, we must have

$$(1.7) \qquad w\left(\sum_{i=1}^{k} b_i \otimes r_i\right) = \inf_{1 \leq i \leq k}\{w_1(b_i) + w_2(r_i)| \ r_i \neq 0\}.$$

We call the unique summand $b_j \otimes r_j$ with minimum value the *leading term* of $\sum b_i \otimes r_i$. It remains to check that formula (1.7) actually defines a valuation on $D_1 \otimes_F D_2$. For $d = \sum_i b_i \otimes r_i$ and $e = \sum_i b_i \otimes s_i \in D_1 \otimes D_2 - \{0\}$ with $e \neq -d$ it is easy to see that $w(d + e) \geq \inf\{w(d), w(e)\}$; hence $w(d + e) = \inf\{w(d), w(e)\}$ if $w(d) \neq w(e)$. For the behavior of w under products, consider first $(b_j \otimes r_j) \cdot (b_k \otimes s_k)$ with $r_j, s_k \neq 0$. Write $b_j b_k = \sum_i c_i b_i$ with the $c_i \in F$. Then,

$$w((b_j \otimes r_j)(b_k \otimes s_k)) = w\left(\sum_i b_i \otimes (c_i r_j s_k)\right)$$
$$= \inf\{w_1(b_i) + v(c_i) + w_2(r_j) + w_2(s_k)| \ c_i \neq 0\}$$
$$= w_1(b_j b_k) + w_2(r_j) + w_2(s_k)$$
$$= w(b_j \otimes r_j) + w(b_k \otimes s_k).$$

Hence, if $de \neq 0$,

$$w(de) = w\left(\sum(b_i \otimes r_i) \cdot \sum(b_j \otimes s_j)\right)$$
$$\geq \inf\{w((b_i \otimes r_i)(b_j \otimes s_j))| \ r_i \neq 0, s_j \neq 0\}$$
$$= \inf\{w(b_i \otimes r_i) + w(b_j \otimes s_j)\} = w(d) + w(e),$$

i.e., $w(de) \geq w(d) + w(e)$. Now, let $b_j \otimes r_j$ be the leading term of d. Then $d = (b_j \otimes r_j) + d'$ where $d' = 0$ or $w(d') > w(b_j \otimes r_j)$. Likewise write $e = (b_k \otimes s_k) + e'$ where $b_k \otimes s_k$ is the leading term of e. Then $de = (b_j \otimes r_j)(b_k \otimes s_k) + \rho$, where $\rho = d'(b_k \otimes s_k) + (b_j \otimes r_j)e' + d'e'$. Now, if $\rho \neq 0, w(\rho) > w((b_j \otimes r_j)(b_k \otimes s_k))$ by what has already been proved. Hence, $\rho \neq -(b_j \otimes r_j)(b_k \otimes s_k)$. Therefore, $de \neq 0$ and $w(de) = w((b_j \otimes r_j)(b_k \otimes s_k)) = w(d) + w(e)$. This shows that $D_1 \otimes D_2$ has

no zero divisors, and that w is a valuation. Since $D_1 \otimes D_2$ is finite-dimensional over D_2, hence Artinian, it must be a division ring. Clearly $\Gamma_{D_1 \otimes D_2} = \Gamma_{D_1} + \Gamma_{D_2}$. Hence,

$$|\Gamma_{D_1 \otimes D_2} : \Gamma_{D_2}| = |\Gamma_{D_1} : \Gamma_{D_1} \cap \Gamma_{D_2}| = |\Gamma_{D_1} : \Gamma_F| = [D_1 : F] = [D_1 \otimes D_2 : D_2],$$

which shows that $D_1 \otimes D_2$ is totally ramified over D_2. □

REMARK. B. Fein has informed us that the special case of the extension lemma with D_1 and D_2 both finite-dimensional and totally ramified over F appeared in the unpublished papers of the late P. Draxl.

2. Symplectic pairings and armatures of algebras. In considering tensor products of symbol algebras or totally ramified division algebras, certain symplectic pairings on finite abelian groups arise naturally. We now summarize the basic facts about such pairings, and then describe the pairing associated with a tensor product of symbol algebras.

Let G be a finite abelian group. We use multiplicative notation for G and write 1_G (or just 1) for its identity element. For $g \in G$, we write $\langle g \rangle$ for the cyclic subgroup generated by g. We say that $\{g_1, \ldots, g_k\}$ is a *base* of G if G is the internal direct product $G = \langle g_1 \rangle \times \cdots \times \langle g_k \rangle$. Let $n_i = \mathrm{ord}(g_i)$, $i = 1, 2, \ldots, k$. If $n_2 | n_1$, $n_3 | n_2, \ldots, n_k | n_{k-1}$ and $n_k > 1$, then $n_1, \ldots, n_k$ are the invariant factors of G, which determine G up to isomorphism. The *rank* of G is the minimum size of a generating set of G, i.e., the number of invariant factors of G.

Let B be a (bimultiplicative) *symplectic pairing* on G, that is, a function

$$B \colon G \times G \to C$$

(where C is a group (written multiplicatively) isomorphic to a subgroup of $\mathbf{Q}/\mathbf{Z}$) such that, for all $g_1, g_2, g, h \in G$,

(i) $B(g_1 g_2, h) = B(g_1, h) B(g_2, h)$;
(ii) $B(h, g_1 g_2) = B(h, g_1) B(h, g_2)$;
(iii) $B(g, g) = 1_C$.

So, $B(g, h) = B(h, g)^{-1}$ for all $g, h \in G$. Note that the image of B is a finite (hence cyclic) subgroup of C. For any subgroup H of G let $H^\perp := \{g \in G | B(g, h) = 1$ for all $h \in H\}$, a subgroup of G. We call $H^\perp$ the *orthogonal dual* of H with respect to B. The *radical* of G, $\mathrm{rad}(G)$, is defined to be $G^\perp$.

Assume B is nondegenerate in the terminology of §1, i.e., $\mathrm{rad}(G) = \{1_G\}$. Then $|\mathrm{im}(B)| = \exp(G)$ and B induces an isomorphism $G \cong \mathrm{Hom}(G, C)$ via $g \mapsto B(g, -)$; also, $\mathrm{Hom}(G, C) \cong \hat{G}$ as C embeds in $\mathbf{Q}/\mathbf{Z}$ and $\exp(G) | |C|$. From the duality for finite abelian groups we have, for any subgroup H of G,

$$(2.1) \qquad G/H^\perp \cong \mathrm{Hom}(H, C) \cong \hat{H} \cong H,$$

so

$$(2.2) \qquad |H^\perp| \cdot |H| = |G| \quad \text{and} \quad H^{\perp\perp} = H.$$

Note that the restriction of B to $H \times H$ is a symplectic pairing on H whose radical is $H \cap H^\perp$. Thus, B induces a nondegenerate symplectic pairing on $H/(H \cap H^\perp)$. If $H \subseteq H^\perp$ we say that H is *totally isotropic* with respect to B. Following the terminology of [**TA₂**] we call a maximal totally isotropic subgroup a *Lagrangian* of B on G. Clearly a subgroup L of G is a Lagrangian if and only if $L = L^\perp$, if and

only if (by (2.2)) L is totally isotropic and $|L| = \sqrt{|G|}$. Different Lagrangians of G need not be isomorphic, nor even have the same rank. But note that if L is any Lagrangian of G, since $G/L \cong L^\perp = L$,

$$(2.3) \qquad\qquad \operatorname{rank} L \leq \operatorname{rank} G \leq 2 \operatorname{rank} L.$$

An extensive analysis of the possible Lagrangians of a group is given in [**TA₃**].

It is well known (cf. [**T₁**, 1.8]) that if B is nondegenerate then G has a *symplectic base* with respect to B, i.e., a base $\{g_1, h_1, g_2, h_2, \ldots, g_k, h_k\}$ such that for all i, j

$$B(g_i, h_i) = c_i, \quad \text{where } \operatorname{ord}(g_i) = \operatorname{ord}(h_i) = \operatorname{ord}(c_i);$$
$$B(g_i, g_j) = B(h_i, h_j) = 1 \quad \text{and, if } i \neq j, \quad B(g_i, h_j) = 1.$$

Such a base can be chosen with $\operatorname{ord}(c_2)| \operatorname{ord}(c_1), \operatorname{ord}(c_3)| \operatorname{ord}(c_2), \ldots, \operatorname{ord}(c_k)|$ $\operatorname{ord}(c_{k-1})$, and $\operatorname{ord}(c_k) > 1$. Then $\operatorname{ord}(c_1), \operatorname{ord}(c_1), \ldots, \operatorname{ord}(c_k), \operatorname{ord}(c_k)$ are the invariant factors of G. The existence of a symplectic base is easy to verify by an orthogonalization process like the one for finding a symplectic base of a symplectic pairing on a vector space. Note that a finite abelian group G admits a nondegenerate symplectic pairing if and only if $G \cong H \times H$ for some group H, and this holds if and only if every invariant factor of G occurs an even number of times.

Suppose G_1 and G_2 are finite abelian groups with symplectic pairings $B_i : G_i \times G_i \to C$, $i = 1, 2$. Their *orthogonal sum* $G_1 \perp G_2$ is the direct product $G_1 \times G_2$ together with the symplectic pairing $B : (G_1 \times G_2) \times (G_1 \times G_2) \to C$ given by $B((g_1, g_2), (h_1, h_2)) = B_1(g_1, h_1) \cdot B_2(g_2, h_2)$. Clearly $\operatorname{rad}(G_1 \perp G_2) = \operatorname{rad}(G_1) \times \operatorname{rad}(G_2)$, so B is nondegenerate if and only if B_1 and B_2 are both nondegenerate.

Let A be an algebra (with 1) over a field F, with $\dim_F A < \infty$. Identify F with $F \cdot 1 \subseteq A$. For $a \in A^*$ let $\tilde{a}$ denote the image aF^* of a in A^*/F^*. Let $\mathcal{A}$ be a subgroup of A^*/F^*, say $\mathcal{A} = \{\tilde{a}_1, \tilde{a}_2, \ldots\}$. Let $F[\mathcal{A}]$ denote the F-subspace of A spanned by $\{a_1, a_2, \ldots\}$. Note that $F[\mathcal{A}]$ depends only on $\mathcal{A}$, independent of the choice of representatives a_i for $\tilde{a}_i \in \mathcal{A}$. Since $\mathcal{A}$ is a group $F[\mathcal{A}]$ is also the sub*algebra* of A generated by $\{a_1, a_2, \ldots\}$.

DEFINITION. For any finite-dimensional F-algebra A, a subgroup $\mathcal{A}$ of A^*/F^* is an *armature* of A if $\mathcal{A}$ is abelian, $|\mathcal{A}| = \dim_F A$, and $F[\mathcal{A}] = A$.

Note that if $\mathcal{A} = \{\tilde{a}_1, \ldots, \tilde{a}_n\}$, the last two conditions for an armature are equivalent to: $\{a_1, \ldots, a_n\}$ is an F-base of A. The notion of an armature was introduced in [**T₁**, §1]. However, the definition given here is different from and slightly more restrictive than the earlier one, but we allow armatures in algebras other than division rings. Here are some examples of armatures (cf. [**T₁**, Example 1.2]).

EXAMPLES 2.4. (a) Let A be a field which is a Kummer extension of F, with $[A : F] < \infty$. Let $l = \exp(\mathcal{G}(A/F))$ and let $S = \{a \in A^* | a^l \in F\}$. Then S/F^* is an armature of A. This is well known from Kummer theory (see, e.g. [**J₁**, pp. 119–123]). Indeed, S/F^* is the only armature of A with exponent dividing l.

(b) Let ω be a primitive nth root of unity in F. For any $a, b \in F^*$, let $A_\omega(a, b; F)$ denote the F-algebra with generators: i, j, and relations: $i^n = a$, $j^n = b$, $ij = \omega ji$. We will call such an algebra a *symbol algebra*, terminology suggested by T.-Y. Lam. (Such an algebra is often called a cyclic algebra or a "cyclic" algebra as in [**Ta**, p. 265]. We prefer to reserve the term cyclic algebra for a crossed-product algebra whose associated Galois field extension is cyclic.) It is well known (cf. [**M**, §15]), that $A_\omega(a, b; F)$ is a central simple F-algebra of dimension n^2 over F; clearly $\{\widetilde{i^r j^s} | 0 \leq r < n, 0 \leq s < n\}$ is an armature of $A_\omega(a, b; F)$.

(c) If $\mathcal{A}$ is an armature of an F-algebra A and $\mathcal{B}$ is any subgroup of $\mathcal{A}$, then $\mathcal{B}$ is an armature of $F[\mathcal{B}]$.

(d) Suppose A_1 and A_2 are F-algebras with respective armatures $\mathcal{A}_1$ and $\mathcal{A}_2$. Then $\mathcal{A} := \{(a \otimes b)F^* \mid \tilde{a} \in \mathcal{A}_1 \text{ and } \tilde{b} \in \mathcal{A}_2\}$ is an armature of $A_1 \otimes_F A_2$. Clearly $\mathcal{A} \cong \mathcal{A}_1 \times \mathcal{A}_2$.

(e) Suppose $\mathcal{A}$ is an armature of an F-algebra A. Let K be any field containing F. Then $\mathcal{A}_K := \{(a \otimes 1)K^* \mid \tilde{a} \in \mathcal{A}\}$ is an armature of the K-algebra $A \otimes_F K$.

Let A be an F-algebra with armature $\mathcal{A}$. As was observed in [$\mathbf{T_1}$, 1.3] there is an associated pairing $B_{\mathcal{A}} : \mathcal{A} \times \mathcal{A} \to F^*$ defined by $B_{\mathcal{A}}(\tilde{a}, \tilde{b}) = [a, b] = aba^{-1}b^{-1}$. Since $F^* \subseteq Z(A)$, $B_{\mathcal{A}}(\tilde{a}, \tilde{b})$ is independent of the choice of representatives a, b for $\tilde{a}, \tilde{b}$, and $B_{\mathcal{A}}(\tilde{a}, \tilde{b}) \in F^*$ as $\mathcal{A}$ is abelian. From the general identity $[r, st] = [r, s]s[r, t]s^{-1}$ it follows that $B_{\mathcal{A}}$ is bimultiplicative. Thus, as $\mathcal{A}$ is finite, the image of $B_{\mathcal{A}}$ is a finite subset of $\mu(F)$. Since $\mu(F)$ embeds in $\mathbf{Q}/\mathbf{Z}$, $B_{\mathcal{A}}$ is a symplectic pairing on $\mathcal{A}$.

LEMMA 2.5. *Let $\mathcal{A}$ be an armature of an F-algebra A, and let $\mathcal{B}$ and $\mathcal{C}$ be any subgroups of $\mathcal{A}$. Then,*

(i) *the centralizer in A of $F[\mathcal{B}]$ is $F[\mathcal{B}^\perp]$;*

(ii) *$F[\mathcal{B}]$ is commutative if and only if $\mathcal{B}$ is totally isotropic;*

(iii) *$Z(A) = F[\mathrm{rad}(\mathcal{A})]$; hence $B_{\mathcal{A}}$ is nondegenerate if and only if $Z(A) = F$;*

(iv) *if $\mathcal{A} = \mathcal{B} \perp \mathcal{C}$, then $A \cong F[\mathcal{B}] \otimes_F F[\mathcal{C}]$.*

PROOF. (Of course $\mathcal{B}^\perp$ and $\mathrm{rad}(\mathcal{A})$ are relative to the symplectic structure on $\mathcal{A}$ given by $B_{\mathcal{A}}$.) Let $\mathcal{A} = \{\tilde{a}_1, \ldots, \tilde{a}_n\}$, so $a_1, \ldots, a_n$ are an F-base of A. Take any $d \in A$ and write $d = \sum c_i a_i$ with $c_i \in F$. For any $\tilde{b} \in \mathcal{B}$, we have

$$db - bd = \sum c_i(a_i b - b a_i) = \sum c_i(1 - B_{\mathcal{A}}(\tilde{b}, \tilde{a}_i))a_i b.$$

Since $b \in A^*$, $a_1 b, \ldots, a_n b$ are F-linearly independent. Hence, d commutes with b if and only if for each i, $c_i = 0$ or $B_{\mathcal{A}}(\tilde{b}, \tilde{a}_i) = 1$. Thus d centralizes $F[\mathcal{B}]$ if and only if d commutes with every b with $\tilde{b} \in \mathcal{B}$, if and only if $c_i = 0$ whenever $\tilde{a}_i \notin \mathcal{B}^\perp$, if and only if $d \in F[\mathcal{B}^\perp]$. This proves (i), and (ii) and (iii) follow immediately.

For (iv), suppose $\mathcal{A} = \mathcal{B} \perp \mathcal{C}$. By this we mean that $\mathcal{A}$ is the direct product of its subgroups $\mathcal{B}$ and $\mathcal{C}$, and $\mathcal{C} \subseteq \mathcal{B}^\perp$. So $F[\mathcal{C}]$ centralizes $F[\mathcal{B}]$, and the inclusions $F[\mathcal{B}] \hookrightarrow A$, $F[\mathcal{C}] \hookrightarrow A$ induce an F-algebra homomorphism $\alpha \colon F[\mathcal{B}] \otimes_F F[\mathcal{C}] \to A$. This map is surjective since $\mathcal{B}$ and $\mathcal{C}$ generate $\mathcal{A}$, which lifts to an F-base of A. By dimension count, α is an isomorphism. $\square$

REMARK 2.6. Let $\mathcal{A}$ be an armature of an F-algebra A. Then $\mathcal{A}$ yields an explicit presentation of A as follows: Let $\tilde{a}_1, \ldots, \tilde{a}_k$ be a base of the abelian group $\mathcal{A}$, let $n_i = \mathrm{ord}(\tilde{a}_i)$, and let $c_i = a_i^{n_i} \in F^*$. ($c_i \in F^*$ as $\tilde{a}_i^{n_i} = 1_{\mathcal{A}}$. The coset $c_i F^{*n_i}$ is uniquely determined by $\tilde{a}_i$, though c_i depends on the choice of representative a_i of $\tilde{a}_i$.) Then A is isomorphic to the F-algebra with generators: $z_1, \ldots, z_k$ and relations: $z_i^{n_i} = c_i$, $z_i z_j = B_{\mathcal{A}}(\tilde{a}_i, \tilde{a}_j) z_j z_i$. For, if A' is the algebra with these generators and relations, then there is a surjective homomorphism $A' \to A$ given by $z_i \mapsto a_i$. This map is an isomorphism, since it is easy to check that $\dim_F A' \leq n_1 \cdots n_k = |\mathcal{A}| = \dim_F A$.

PROPOSITION 2.7. *Let A be a finite-dimensional F-algebra with $Z(A) = F$. Then the following statements are equivalent.*

(i) *A has an armature.*

(ii) *A is isomorphic to a tensor product of symbol algebras over F.*

PROOF. (ii)$\Rightarrow$(i) is immediate from Examples 2.4(b) and (d).

(i)$\Rightarrow$(ii) Let $\mathcal{A}$ be an armature on A, and $B_{\mathcal{A}}$ the associated symplectic pairing. Since $Z(A) = F$ the lemma says that $B_{\mathcal{A}}$ is nondegenerate. So, $\mathcal{A}$ has a symplectic base $\{\tilde{a}_1, \tilde{b}_1, \ldots, \tilde{a}_k, \tilde{b}_k\}$. Let $\omega_i = B_{\mathcal{A}}(\tilde{a}_i, \tilde{b}_i) \in \mu(F)$, and let $n_i = \operatorname{ord}(\omega_i) = \operatorname{ord}(\tilde{a}_i) = \operatorname{ord}(\tilde{b}_i)$. Let $c_i = a_i^{n_i} \in F^*$ and $d_i = b_i^{n_i} \in F^*$, and let $\mathcal{A}_i$ be the subgroup of $\mathcal{A}$ generated by $\tilde{a}_i, \tilde{b}_i$. Then the presentation given above shows that $F[\mathcal{A}_i] \cong A_{\omega_i}(c_i, d_i; F)$. Since $\mathcal{A} = \mathcal{A}_1 \perp \cdots \perp \mathcal{A}_k$ the lemma says $A \cong \bigotimes_{i=1}^{k} A_{\omega_i}(c_i, d_i; F)$. $\square$

COROLLARY 2.8 (cf. [$\mathbf{T_1}$, Proposition 1.9]).   *If A is an F-algebra with armature $\mathcal{A}$ and $Z(A)$ is a field, then A is isomorphic to a tensor product of symbol algebras over $Z(A)$.*

PROOF. Let $K = Z(A) = F[\operatorname{rad}(\mathcal{A})]$. Let $\mathcal{A}'$ be the image of $\mathcal{A}$ in the natural homomorphism $g: A^*/F^* \to A^*/K^*$. Then $K[\mathcal{A}'] = F[\mathcal{A}] = A$. Since $|\operatorname{rad}(\mathcal{A})| = [K : F]$ and the inverse images in A of elements of $\mathcal{A}$ are F-linearly independent, we have $\ker(g) \cap \mathcal{A} = K^* \cap \mathcal{A} = \operatorname{rad}(\mathcal{A})$. Hence,

$$|\mathcal{A}'| = |\mathcal{A}|/|\operatorname{rad}(\mathcal{A})| = (\dim_F A)/[K : F] = \dim_K A.$$

Therefore, $\mathcal{A}'$ is an armature of A as a K-algebra, and Proposition 2.7 yields the corollary. $\square$

Now, suppose A is a *commutative* F-algebra with armature $\mathcal{A}$, and let l be a positive integer with $\exp(\mathcal{A})|l$. Then there is a well-defined group homomorphism $f: \mathcal{A} \to F^*/F^{*l}$ given by $\tilde{a} \mapsto a^l F^{*l}$. Let $\tilde{a}_1, \ldots, \tilde{a}_k$ be a base of $\mathcal{A}$, and let $\operatorname{ord}(\tilde{a}_i) = n_i$ and $a_i^{n_i} = c_i \in F^*$. In the extreme cases for f we obtain a nice description of A.

LEMMA 2.9.   *Suppose $\mu_l \subseteq F$. Then,*
(a) *if f is injective, then $A \cong F(c_1^{1/n_1}, \ldots, c_k^{1/n_k})$, a field;*
(b) *if $\ker(f) = \mathcal{A}$, then $A \cong F \oplus F \oplus \cdots \oplus F$.*

PROOF. (a) From the presentation of A in (2.6) there is clearly a surjective F-algebra homomorphism $h: A \to F(c_1^{1/n_1}, \ldots, c_k^{1/n_k})$ given by $a_i \mapsto c_i^{1/n_i}$. By Kummer theory $[F(c_1^{1/n_1}, \ldots, c_k^{1/n_k}) : F]$ equals the order of the subgroup of F^*/F^{*l} generated by $\{c_1^{l/n_1}, \ldots, c_k^{l/n_k}\}$. This group is $\operatorname{im}(f)$, so assuming f is injective, its order is $|\mathcal{A}| = \dim_F A$. Thus a dimension comparison shows h is an isomorphism.

(b) Suppose $\ker(f) = \mathcal{A}$. Then, for each i, $c_i^{l/n_i} = a_i^l \in F^{*l}$. Say $c_i^{l/n_i} = d_i^l$, for some $d_i \in F^*$. Then $c_i = \omega_i d_i^{n_i}$ for some $\omega_i \in \mu_{l/n_i}(F)$. Since $\mu_l \subseteq F$, ω_i has an n_ith root in F, and we can write $c_i = e_i^{n_i}$ for some $e_i \in F^*$. From the presentation given in (2.6) for A we have $A \cong F[z_1, \ldots, z_k]/(z_1^{n_1} - c_1, \ldots, z_k^{n_k} - c_k)$, where the z_i are commuting indeterminates. Since each $z_i^{n_i} - c_i = z_i^{n_i} - e_i^{n_i}$ factors into n_i distinct linear factors, A is a direct sum of copies of F. $\square$

3. Totally ramified tame division algebras. Let (D, v) be any valued division algebra which is tame and totally ramified over its center F. (Recall that this means: $|\Gamma_D : \Gamma_F| = [D : F] < \infty$, so $\overline{D} = \overline{F}$, and $\operatorname{char}(\overline{F}) \nmid [D : F]$.) The finite abelian group Γ_D/Γ_F is an important invariant of D, which we call the *relative*

value group of D with respect to v. We will define a natural pairing on Γ_D/Γ_F via commutators, and will show that this pairing carries substantial information about the structure of D.

Define a function $g: D^* \times D^* \to \overline{F}^*$ by

$$g(d,e) = \overline{ded^{-1}e^{-1}}.$$

This function is well-defined as Γ_D is abelian, so all commutators lie in U_D. Since any inner automorphism of D maps the valuation ring V_D to itself and induces the trivial automorphism on $\overline{D} = \overline{F}$, we have

$$g(d, e_1 e_2) = \overline{de_1 e_2 d^{-1} e_2^{-1} e_1^{-1}} = \overline{de_1 d^{-1} e_1^{-1}} \cdot \overline{e_1 (de_2 d^{-1} e_2^{-1}) e_1^{-1}}$$
$$= g(d, e_1) g(d, e_2).$$

Thus, g is multiplicative in the second component, likewise in the first component, and is clearly symplectic. Note that the radical of g contains F^* and the unit group U_D. Because $D^*/F^* U_D \cong \Gamma_D/\Gamma_F$, g induces a (bilinear) symplectic pairing

$$C_D: (\Gamma_D/\Gamma_F) \times (\Gamma_D/\Gamma_F) \to \overline{F}^*,$$

which we call the *canonical pairing* of v on D.

PROPOSITION 3.1. *For any valued division algebra (D,v) tame and totally ramified over its center F, the canonical pairing C_D on Γ_D/Γ_F is nondegenerate. Consequently, $\mu_l \subseteq \overline{F}$, where $l = \exp(\Gamma_D/\Gamma_F)$, and $\Gamma_D/\Gamma_F \cong H \times H$ for some finite abelian group H.*

PROOF. Assume first that (F, v) is Henselian and $\mu_l \subseteq F$, so $\mu_l \subseteq \overline{F}$. Take any $a \in D^*$ with $v(a) \notin \Gamma_F$. We show that $\overline{v(a)} := v(a) + \Gamma_F$ is not in $\mathrm{rad}(\Gamma_D/\Gamma_F)$. Let $n = \mathrm{ord}(\overline{v(a)})$ in Γ_D/Γ_F, so $n|l$. Then there is a $b \in F^*$ with $v(b) = v(a^n)$ and, as $\overline{D} = \overline{F}$, a $c \in U_F$ with $\overline{a^n b^{-1}} = \bar{c}$ in $\overline{D}$. So $\overline{a^n b^{-1} c^{-1}} = 1$ in $\overline{F(a)}$. Since $F(a)$ is Henselian and $\mu_n \subseteq F(a)$ every 1-unit of $F(a)$ has an nth root in $F(a)$; so there is a $u \in F(a)$ with $v(u) = 0$ and $u^n = a^n b^{-1} c^{-1}$. Let $a' = au^{-1}$. Then $a'^n = bc \in F^*$ with $v(a') = v(a)$. Also $[F(a') : F] \geq |\Gamma_{F(a')} : \Gamma_F| \geq \mathrm{ord}\,\overline{v(a')} = n$. Hence, by Kummer theory, for any primitive nth root of unity $\omega \in F^*$ there is an F-automorphism τ of $F(a')$ with $\tau(a') = a'\omega$. The Skolem-Noether theorem shows there is a $d \in D^*$ with $da'd^{-1} = \tau(a') = a'\omega$. Then $C_D(\overline{v(d)}, \overline{v(a')}) = \overline{da'd^{-1}a'^{-1}} = \overline{\omega}$. Because $\mathrm{char}(\overline{F}) \nmid n$, $\mathrm{ord}(\overline{\omega}) = \mathrm{ord}(\omega) = n$, so $\overline{\omega} \neq 1$. Thus, $\overline{v(a)} = \overline{v(a')} \notin \mathrm{rad}(\Gamma_D/\Gamma_F)$, proving the nondegeneracy of C_D.

Now drop the restrictions on F. Let (F_h, v) be a Henselization of (F, v) and let $K = F_h(\lambda)$ where λ is a primitive lth root of unity. Because F_h is Henselian, v extends uniquely to K; further, K is Henselian and an inertial extension of F_h, which is an immediate extension of F. Hence, $\Gamma_K = \Gamma_{F_h} = \Gamma_F$. By the extension Lemma 1.6, $D \otimes_F K$ is a division algebra totally ramified over K, and $\Gamma_{D \otimes K} = \Gamma_D$. When we identify Γ_D/Γ_F with $\Gamma_{D \otimes K}/\Gamma_K$, the canonical pairing for D is the same as the pairing for $D \otimes_F K$, which we have seen to be nondegenerate. The nondegeneracy assures that the image of C_D contains all lth roots of unity and, as noted in §2, that Γ_D/Γ_F has the form $H \times H$. $\square$

The fact that $\mu_l \subseteq \overline{F}$ was noted in the Henselian case in [**PY**, Proposition 5].

REMARKS 3.2. (a) If D is a division algebra with valuation tame and totally ramified over its center F, then the nondegenerate pairing assures that Γ_D/Γ_F has

even rank as an abelian group. Since $\dim_{\mathbf{Q}}(\Gamma_F \otimes_{\mathbf{Z}} Q) \geq \mathrm{rank}(\Gamma_D/\Gamma_F) \geq 2$ if $D \neq F$, we must have $\Gamma_F \neq \mathbf{Z}$. Thus, discrete (rank 1) valued fields do not admit totally ramified division algebras. This fact may explain why totally ramified division algebras were not studied many years earlier.

(b) In order to define the canonical pairing on D, it suffices to have a valuation on D with $\overline{D} = \overline{Z(D)}$. This does not lead to a generalization of Proposition 3.1, however, since it can be shown that if D is tame over $Z(D)$ and $\overline{D} = \overline{Z(D)}$, then D is totally ramified over $Z(D)$.

The next proposition generalizes a result of the first author [$\mathbf{T_1}$, Theorem 3.6] on the possible cyclic decompositions of certain twisted iterated Laurent series division algebras. Note that its proof uses the existence of the canonical pairing, but not its nondegeneracy.

PROPOSITION 3.3. *Let (D, v) be a valued division algebra with D tame and totally ramified over $Z(D) = F$. If $\mathcal{A}$ is an armature of D as an F-algebra then the map $\bar{v}: \mathcal{A} \to \Gamma_D/\Gamma_F$ induced by v is an isomorphism. Consequently, if $D \cong A_1 \otimes_F \cdots \otimes_F A_k$, where the A_i are symbol algebras with $\dim_F A_i = n_i^2 > 1$ and $n_2|n_1,\ n_3|n_2,\ldots,n_k|n_{k-1}$, then $n_1, n_1, n_2, n_2, \ldots, n_k, n_k$ are uniquely determined as the invariant factors of Γ_D/Γ_F.*

PROOF. The armature pairing $B_{\mathcal{A}}$ on $\mathcal{A}$ (see §2) and the canonical pairing C_D are both based on commutators. Thus, it is easy to see that the following diagram is commutative:

(3.4)
$$
\begin{array}{ccc}
\mathcal{A} \times \mathcal{A} & \xrightarrow{\quad B_{\mathcal{A}} \quad} & \mu(F) \\
{\scriptstyle (\bar{v}, \bar{v})} \downarrow & & \downarrow {\scriptstyle \rho} \\
(\Gamma_D/\Gamma_F) \times (\Gamma_D/\Gamma_F) & \xrightarrow{\quad C_D \quad} & \mu(\overline{F})
\end{array}
$$

where $\rho: V_F \to \overline{F}$ is the natural projection. (This diagram is analogous to (1.5) for field extensions.) Because $B_{\mathcal{A}}$ is nondegenerate by (2.5)(iii) and ρ is injective on roots of unity of order prime to $\mathrm{char}(\overline{F})$, $\bar{v}$ must be injective. Hence $\bar{v}$ is an isomorphism, as $|\mathcal{A}| = [D : F] = |\Gamma_D : \Gamma_F|$. If D is a tensor product of symbol algebras of dimensions n_i^2, we saw in §2 that D has an armature $\mathcal{A}$ with $\mathcal{A} \cong \prod_{i=1}^{k}(\mathbf{Z}/n_i\mathbf{Z}) \times (\mathbf{Z}/n_i\mathbf{Z})$. The rest of the corollary follows immediately from this. $\square$

Given a commutative diagram such as (3.4) (with a canonical isomorphism between $\mathrm{im}(B_{\mathcal{A}})$ and $\mathrm{im}(C_D)$) we say that the isomorphism $\bar{v}$ is an *isometry* between the nondegenerate pairings $B_{\mathcal{A}}$ and C_D.

We recall from [$\mathbf{JW}$] a useful device for constructing totally ramified division algebras. Let A be a central simple F-algebra, where F is a field with valuation v. Let Δ be the divisible hull of Γ_F. Suppose A has an armature $\mathcal{A}$. Then there is a well-defined function $\bar{w}: \mathcal{A} \to \Delta/\Gamma_F$ given by $aF^* \mapsto (1/n)v(a^n) + \Gamma_F$, where $n = |\mathcal{A}| = \dim_F A$.

PROPOSITION 3.5. *The map $\bar{w}$ just defined is a group homomorphism. Suppose $\bar{w}$ is injective. Then A is a division algebra and v extends to a valuation on A with A totally ramified over F; Γ_A is the subgroup of Δ such that $\Gamma_A/\Gamma_F = \mathrm{im}(\bar{w})$; and $\bar{w} = \bar{v}$ (where $\bar{v}$ is the map in Proposition 3.3).*

PROOF. This is a special case of [**JW**, Theorem 2.5], with $L = F$. The valuation on A is defined by taking inverse images $a_1, \ldots, a_n \in A^*$ of the elements $a_i F^*$ of $\mathcal{A}$ and setting

$$v\left(\sum c_i a_i\right) = \inf\{\tfrac{1}{n}v(a_i^n) + v(c_i) | c_i \neq 0\}$$

for any $c_i \in F$ not all zero. $\square$

If (F, v) is Henselian and $\overline{F}$ is separably closed, we will describe the structure of A even when $\bar{w}$ is not injective—see Theorem 4.3 below.

EXAMPLES 3.6. (a) Let $n_1, \ldots, n_k$ be any positive integers, and let F_0 be any field containing primitive n_ith roots of unity ω_i, $i = 1, 2, \ldots, k$. Let $x_1, \ldots, x_{2k}$ be $2k$ independent indeterminantes over F_0, and let $F = F_0(x_1, \ldots, x_{2k})$. Let $F' = F_0((x_1)) \cdots ((x_{2k}))$ be the iterated Laurent power series field; that is, $F' = F_{2k}$ where for each m, $F_m = F_{m-1}((x_m)) := \{\sum_{l=n}^{\infty} c_l x_m^l | n \in \mathbf{Z} \text{ and all } c_l \in F_{m-1}\}$. There is a standard valuation $v \colon F'^* \to \mathbf{Z}^{2k}$ given by

$$v\left(\sum_{l_1} \cdots \sum_{l_{2k}} c_{l_1 \cdots l_{2k}} x_1^{l_1} \cdots x_{2k}^{l_{2k}}\right) = \inf\{(l_1, \ldots, l_{2k}) | \ c_{l_1 \cdots l_{2k}} \neq 0\},$$

where $\mathbf{Z}^{2k}$ is given the right-to-left lexicographical ordering. In this ordering $(l_1, \ldots, l_{2k}) < (m_1, \ldots, m_{2k})$ just when there is a q with $l_q < m_q$ and $l_r = m_r$ for $q < r \leq 2k$. With respect to v we have $\Gamma_{F'} = \mathbf{Z}^{2k}$ and $\overline{F'} = F_0$, and, as is well known (cf. [**Ri**, p. 77, Proposition 4; p. 198, Theorem 4]), F' is Henselian. This valuation restricts to a valuation on F, and (F', v) is clearly an immediate extension of (F, v).

Now, let

$$D = \bigotimes_{m=1}^{k} {}_F \, A_{\omega_m}(x_{2m-1}, x_{2m}; F),$$

and let $i_m, j_m \in D^*$ be generators of $A_{\omega_m}(x_{2m-1}, x_{2m}; F)$ with $i_m^{n_m} = x_{2m-1}$, $j_m^{n_m} = x_{2m}$, $i_m j_m = \omega_m j_m i_m$. Then the subgroup $\mathcal{A}$ of D^*/F^* generated by the $\tilde{i}_m, \tilde{j}_m, 1 \leq m \leq k$, is an armature of D, and $\{\tilde{i}_1, \tilde{j}_1, \ldots, \tilde{i}_m, \tilde{j}_m\}$ is a symplectic base of $\mathcal{A}$ with respect to the armature pairing $B_{\mathcal{A}}$. Here $\Delta = \mathbf{Q}^{2k}$ and the map $\bar{w}$ of Proposition 3.5 sends

$$i_1 F^* \mapsto (1/n_1, 0, \ldots, 0) + \Gamma_F, \ldots, j_k F^* \mapsto (0, \ldots, 0, 1/n_k) + \Gamma_F.$$

Thus $|\bar{w}(\mathcal{A})| \geq |\mathcal{A}|$, which assures that $\bar{w}$ is injective. Hence, the proposition shows that D is a division algebra and v extends to D so that D is totally ramified over F. We have

$$\Gamma_D = \prod_{m=1}^{k} (\mathbf{Z}/n_m) \times (\mathbf{Z}/n_m), \quad \text{so } \Gamma_D/\Gamma_F \cong \prod_{m=1}^{k} (\mathbf{Z}/n_m\mathbf{Z}) \times (\mathbf{Z}/n_m\mathbf{Z}).$$

Likewise, if L is any field, $F \subseteq L \subseteq F'$, then $D \otimes_F L$ has a valuation extending v on L with $\Gamma_{D \otimes L} = \Gamma_D$, and $D \otimes L$ totally ramified over L. This can be seen in the same way as for D since $\Gamma_L = \Gamma_F$, or by using the extension Lemma 1.6. The division algebras of the type $D \otimes_F F'$ were used by Amitsur in his construction of noncrossed products; the relative value group $\Gamma_{D \otimes F'}/\Gamma_{F'}$ gives strong control over the possible Galois groups of maximal subfields of $D \otimes F'$ (see the comments in [**JW**, (2.3), (2.8)]). Other information about the structure of D determined by Γ_D/Γ_F is collected in (4.7) below.

(b) To be more specific, suppose F_0 contains a primitive 4th root of unity ω, and let $k = n_1 = n_2 = 2$. So, D is the tensor product of two quaternion algebras: $D = ((x_1, x_2)/F) \otimes_F ((x_3, x_4)/F)$, where $F = F_0(x_1, \ldots, x_4)$. It is immediate from Proposition 3.3 that D is not a symbol algebra, i.e., D has no maximal subfield cyclic over F. For, any armature A of D must have $A \cong \Gamma_D/\Gamma_F \cong (\mathbf{Z}/2\mathbf{Z})^4$, but if D were a symbol algebra, it would have an armature of rank 2. We will show later (see (4.7)) that D is not split by any cyclic Galois extension of F. Draxl shows in $[\mathbf{D_1}$, pp. 168–169] that for the corresponding power series algebra D' of D $(D' = D \otimes_F F')$, $SK_1(D') \neq 1$; that is, there is an $a \in D'^*$ with $\mathrm{Nrd}(a) = 1$ (where Nrd is the reduced norm) but a is not in the commutator group $[D'^*, D'^*]$. This can be seen very easily for D' and also for D using the canonical pairing: We take $a = \omega \in F_0$. Then, as $[D : F] = 4^2$, $\mathrm{Nrd}(\omega) = \omega^4 = 1$. But the construction of the canonical pairing shows that the image of $[D^*, D^*]$ in $\overline{D}$ lies in $\mu_l(\overline{F})$, where $l = \exp(\Gamma_D/\Gamma_F) = 2$. Since $\overline{\omega}$ is a primitive 4th root of unity in $\overline{F}$, ω cannot lie in $[D^*, D^*]$. Note that the nondegeneracy of the canonical pairing was not needed for this argument. This is a remarkably easy example, considering that it was an open question (the "Tannaka-Artin problem") for some decades whether any finite-dimensional division algebra D existed with $SK_1(D) \neq 1$. (For more on SK_1 of division algebras see $[\mathbf{DK}]$ and $[\mathbf{P}]$.)

When conditions are imposed on the valuation on the center, more can be said about the structure of a valued division algebra D that is tame and totally ramified over $Z(D)$. We consider here the case where $(Z(D), v)$ is Henselian, and in §4 the case where $(Z(D), v)$ is Henselian and $\overline{Z(D)}$ is separably closed. The following theorem of Draxl $[\mathbf{D_2}$, Theorem 1] makes the Henselian case quite tractable.

DECOMPOSITION THEOREM 3.7 (DRAXL). *Let (D, v) be a valued division algebra which is tame and totally ramified over $F = Z(D)$. If (F, v) is Henselian, then D is isomorpic to a tensor product of symbol algebras.*

It is a nice observation of D. Saltman (unpublished) that if the absolute Galois group of a field K is abelian, then every finite-dimensional K-central division algebra is isomorphic to a tensor product of symbol algebras. The "local" version of this observation says that if D is an F-central division algebra over any field F and if every subfield L of D with $L \supseteq F$ is a Kummer extension of F, then D is isomorphic to a tensor product of symbol algebras. Note that Draxl's decomposition theorem is an instance of the local Saltman result. For, with D and F as in (3.7) we have seen in (3.1) that $\mu_l \subseteq \overline{F}$; then (1.4)(iii) yields the Kummer condition on the intermediate fields.

For certain Malcev-Neumann division algebras tame and totally ramified over their (Henselian valued) centers, the first author and Amitsur gave in $[\mathbf{TA_2}$, Theorem 4.2] a full classification of the subfields of the division algebras, which we now generalize:

THEOREM 3.8. *Let (D, v) be a valued division algebra tame and totally ramified over $Z(D) = F$. Assume F is Henselian with respect to v. Then there is a 1-1 correspondence between the isomorphism classes of F-subalgebras A of D and the subgroups Γ_A/Γ_F of Γ_D/Γ_F. In this correspondence the centralizer of an algebra A corresponds to $(\Gamma_A/\Gamma_F)^\perp$ relative to the canonical pairing C_D. Hence, A is a field if and only if Γ_A/Γ_F is totally isotropic.*

PROOF. As noted above, we have $\mu_l \subseteq F$, where $l = \exp(\Gamma_D/\Gamma_F)$. By Draxl's Theorem 3.7 and by (2.4)(b) and (d) above D has an armature $\mathcal{D}$, say $\mathcal{D} = \{\tilde{d}_1, \ldots, \tilde{d}_n\}$, where $\tilde{d}_i = d_i F^*$. We have seen that the map of (3.3) $\bar{v}\colon \mathcal{D} \to \Gamma_D/\Gamma_F$ is an isomorphism. For any F-subalgebra A of D, let $\mathcal{B} = \bar{v}^{-1}(\Gamma_A/\Gamma_F) \subseteq \mathcal{D}$. Observe that $\Gamma_{F[\mathcal{B}]} = \Gamma_A$. We will show that $A \cong F[\mathcal{B}]$.

First, take any $a \in A^*$, and suppose $a^r \in F^*$, where $r = \text{ord}(v(a) + \Gamma_F)$ in Γ_D/Γ_F. There is a unique $\tilde{d}_j \in \mathcal{D}$ with $\bar{v}(\tilde{d}_j) = v(a) + \Gamma_F$. Then $\tilde{d}_j \in \mathcal{B}$, and we claim that $a^r = s^r d_j^r$ for some $s \in F^*$. To see this, first note that since $\overline{D} = \overline{F}$ and $v(a) \equiv v(d_j) \pmod{\Gamma_F}$ there exist $e \in F^*$ and $m \in D^*$ with $v(m) > 0$ such that $d_j = ae(1 + m)$. Because $\text{ord}(\tilde{d}_j)$ (in $\mathcal{D}$) $= \text{ord}(\bar{v}(\tilde{d}_j))$ (in Γ_D/Γ_F) $= r$, $d_j^r \in F^*$. But

$$d_j^r = e^r a(1 + m)a^{-1}a^2(1 + m)a^{-2} \cdots a^r(1 + m)a^{-r} \cdot a^r.$$

Since $a^k(1 + m)a^{-k} = 1 + a^k m a^{-k} \in 1 + M_D$ for every k, we have $d_j^r e^{-r} a^{-r} \in (1 + M_D) \cap F = 1 + M_F$. The claim then follows, since $1 + M_F \subseteq F^{*r}$ as F is Henselian and $\text{char}(\overline{F}) \nmid r$.

In verifying $A \cong F[\mathcal{B}]$, we first consider some special cases. Suppose A is a field. Because A is commutative Γ_A/Γ_F is totally isotropic with respect to C_D. Hence (cf. (3.3)) $\mathcal{B}$ must be totally isotropic with respect to the armature pairing $B_{\mathcal{D}}$; so $F[\mathcal{B}]$ is commutative. Since A is totally ramified over F, which is Henselian and contains enough roots of unity, A is a Kummer extension of F. Hence, A has an armature $\mathcal{A} = \{a \in A^* \mid a^l \in F^*\} \subseteq A^*/F^*$, where $l = \exp(\mathcal{G}(A/F))$. As noted in §1, $\mathcal{A}$ maps isomorphically to Γ_A/Γ_F. Since also $\mathcal{B} \cong \Gamma_A/\Gamma_F$, these isomorphisms combine to give an isomorphism $g\colon \mathcal{A} \to \mathcal{B}$. Take a base $\{\tilde{a}_1, \ldots, \tilde{a}_k\}$ of the abelian group $\mathcal{A}$, and let $\tilde{b}_i = g(\tilde{a}_i)$. The claim proved above shows that we can choose representatives a_i of $\tilde{a}_i$ and b_i of $\tilde{b}_i$ so that $a_i^{r_i} = b_i^{r_i} \in F$, where $r_i = \text{ord}(\tilde{a}_i) = \text{ord}(\tilde{b}_i)$. Hence, by the presentation given in (2.6), $A = F[\mathcal{A}] \cong F[z_1, \ldots, z_k]/(z_1^{r_1} - a_1^{r_1}, \ldots, z_k^{r_k} - a_k^{r_k}) \cong F[\mathcal{B}]$, as desired.

Instead of assuming A is a field, now assume that $Z(A) = F$. By Draxl's decomposition Theorem 3.7, A has an armature as an F-algebra, which again we call $\mathcal{A}$. From the isomorphism $\mathcal{A} \to \Gamma_A/\Gamma_F$ (see (3.3)) we obtain as before an isomorphism $g\colon \mathcal{A} \to \mathcal{B}$. Let $B_{\mathcal{A}}$ and $B_{\mathcal{B}}$ be the armature pairings for $\mathcal{A}$ and $\mathcal{B}$. For any $\tilde{a}, \tilde{c} \in \mathcal{A}$, $B_{\mathcal{A}}(\tilde{a}, \tilde{c})$ is a root of unity whose image in $\overline{F}$ is determined by the images of $\tilde{a}$ and $\tilde{c}$ in Γ_D/Γ_F. Therefore the map g is an isometry between $B_{\mathcal{A}}$ and $B_{\mathcal{B}}$. Choose a base $\{\tilde{a}_1, \ldots, \tilde{a}_k\}$ of $\mathcal{A}$. As in the field case we can choose representatives a_i of $\tilde{a}_i$ in A and b_i of $g(\tilde{a}_i)$ in $F[\mathcal{B}]$ so that $a_i^{r_i} = b_i^{r_i}$, where $r_i = \text{ord}(\tilde{a}_i) = \text{ord}(g(\tilde{a}_i))$. Then the presentation (2.6) of $F[\mathcal{A}]$ using the a_i is the same as the presentation of $F[\mathcal{B}]$ using the b_i. Thus, $A = F[\mathcal{A}] \cong F[\mathcal{B}]$.

Finally, for the general case we drop the restrictions on A. Let $E = Z(A)$, and let $\mathcal{E} = \bar{v}^{-1}(\Gamma_E/\Gamma_F) \subseteq \mathcal{A}$. By the field case $E \cong F[\mathcal{E}]$. By Skolem-Noether, we may, after conjugating A by an inner automorphism, assume $E = F[\mathcal{E}]$. Let $\mathcal{E}^\perp$ be the orthogonal dual of $\mathcal{E}$ with respect to $B_{\mathcal{D}}$, and let $D' = F[\mathcal{E}^\perp]$, which by (2.5) is the centralizer of E in D. Observe that $\mathcal{E} \subseteq \mathcal{B} \subseteq \mathcal{E}^\perp$; the latter inclusion holds because $B_{\mathcal{D}}$ is isometric to C_D, and $C_D(\Gamma_{F[\mathcal{B}]}/\Gamma_F, \Gamma_E/\Gamma_F) = C_D(\Gamma_A/\Gamma_F, \Gamma_E/\Gamma_F) = 1$. Let $h\colon \mathcal{E}^\perp \to D'^*/E^*$ be the map given by $cF^* \mapsto cE^*$. Note that $\ker(h) = \mathcal{E}$ since $(E^*/F^*) \cap \mathcal{D} = \mathcal{E}$. Therefore, as $E[h(\mathcal{E}^\perp)] = F[\mathcal{E}^\perp] = D'$ and $[D' : E] = |h(\mathcal{E}^\perp)|$,

$h(\mathcal{E}^{\perp})$ is an armature of D' as an E-algebra. Furthermore, the image of $h(\mathcal{B})$ in $\Gamma_{D'}/\Gamma_E$ is Γ_A/Γ_E. So, by the preceding paragraph $A \cong E[h(\mathcal{B})] = F[\mathcal{B}]$, as desired.

Since $\mathcal{B} = \bar{v}^{-1}(\Gamma_A/\Gamma_F)$, we have shown that A is determined up to isomorphism by Γ_A/Γ_F. Furthermore, if S is any subgroup fo Γ_D/Γ_F, $F[\bar{v}^{-1}(S)]$ is a subalgebra of D and $\Gamma_{F[\bar{v}^{-1}(S)]}/\Gamma_F = S$. This gives the desired one-to-one correspondence. The rest of the theorem is immediate from (2.5)(i). $\square$

4. Division algebras over strictly Henselian fields. Let (F, v) be a *strictly Henselian* valued field, i.e., suppose v is a Henselian valuation on F with $\overline{F}$ separably closed. If D is a tame F-central division algebra, Draxl's analogue to Ostrowski's Theorem ((1.2) above) implies that D is totally ramified over F with respect to the unique extension of v to D. We will show, in fact, that D is determined up to isomorphism by Γ_D and the canonical pairing Γ_D/Γ_F—see Proposition 4.2 below. The main result of this section is a method for determining the underlying division algebra of a tame central simple F-algebra.

Before considering division algebras over F, let us recall the classification of tame field extensions of F. As usual, Δ denotes the divisible hull of Γ_F.

PROPOSITION 4.1. *Suppose (F, v) is a strictly Henselian valued field. For any tame field extension L of F, L is totally ramified and Galois over F with $\mathcal{G}(L/F) \cong (\widehat{\Gamma_L/\Gamma_F})$. The map $L \mapsto \Gamma_L$ gives a one-to-one inclusion-preserving correspondence between the tame field extensions of F and all the subgroups Γ of Δ with $\Gamma_F \subseteq \Gamma$ and $\mathrm{char}(\overline{F}) \nmid |\Gamma : \Gamma_F| < \infty$.*

Proposition 4.1 follows easily from (1.4) above, in light of Ostrowski's theorem (the commutative version of (1.2)—see, e.g., [**E**, (20.21)]). The proposition says, in particular, that L is determined up to isomorphism by Γ_L. For a tame F-central division algebra D it is easy to see that Γ_D does not completely determine D. We show that the further information about the noncommutative multiplication needed to specify D completely is carried by the canonical pairing on Γ_D/Γ_F defined in §3.

PROPOSITION 4.2. *Let (F, v) be a strictly Henselian valued field. Let Γ be a subgroup of the divisible hull of Γ_F with $\Gamma_F \subseteq \Gamma$ and $\mathrm{char}(\overline{F}) \nmid |\Gamma : \Gamma_F| < \infty$. Suppose Γ/Γ_F admits a nondegenerate symplectic pairing B into $\mu(\overline{F})$. Then there is a unique (up to isomorphism) division algebra D tame over $F = Z(D)$ with $\Gamma_D = \Gamma$ and canonical pairing $C_D = B$. If (Γ', B') is another such subgroup and nondegenerate symplectic pairing, then the corresponding D' is isomorphic to a subalgebra of D if and only if $\Gamma' \subseteq \Gamma$ and B' is the restriction of B to Γ'/Γ_F.*

PARTIAL PROOF. For the moment we show how D can be constructed from Γ and B. Let $\{\overline{\alpha}_1, \overline{\beta}_1, \ldots, \overline{\alpha}_k, \overline{\beta}_k\}$ be a symplectic base of Γ/Γ_F with respect to B. For $1 \le i \le k$, let $\overline{\omega}_i = B(\overline{\alpha}_i, \overline{\beta}_i) \in \mu(\overline{F})$, let $n_i = \mathrm{ord}(\overline{\alpha}_i) = \mathrm{ord}(\overline{\beta}_i) = \mathrm{ord}(\overline{\omega}_i)$, let ω_i be the unique inverse image of $\overline{\omega}_i$ in $\mu(F)$ with $\mathrm{ord}(\omega_i) = n_i$. Pick any inverse images $\alpha_i, \beta_i \in \Gamma$ of $\overline{\alpha}_i, \overline{\beta}_i$, and pick any $a_i, b_i \in F^*$ with $v(a_i) = n_i\alpha_i$ and $v(b_i) = n_i\beta_i$. Then set

$$D = \bigotimes_{i=1}^{k} {}_F A_{\omega_i}(a_i, b_i; F).$$

Let $\mathcal{D}$ be the obvious armature of D (cf. 2.4(b) and (d)). We have $|\mathcal{D}| = |\Gamma/\Gamma_F|$ and clearly $\Gamma/\Gamma_F \subseteq \bar{w}(\mathcal{D})$, where $\bar{w}$ is the map of Proposition 3.5. That proposition

shows D is a division algebra with $\Gamma_D = \Gamma$ and $C_D = B$, as desired. The remainder of the proof is deferred until after the proof of the next theorem.

Now, let A be a tame central simple F-algebra. By Wedderburn's theorem A is a matrix algebra, $A \cong M_n(D)$ for some division algebra D with $Z(D) = F$ and D tame over F. This D is unique up to isomorphism; we call it the *underlying division algebra* of A. Over an arbitrary base field it seems to be a hopelessly difficult task to identify the underlying division algebra of a central simple F-algebra. Nonetheless, if F is strictly Henselian and A is tame we will now give a method for finding D, exploiting Draxl's decomposition Theorem 3.7 to see that A has an armature.

THEOREM 4.3. *Let (F, v) be a strictly Henselian valued field, and let A be any central simple F-algebra with $\mathrm{char}(\overline{F}) \nmid \dim_F A$. Let $\mathcal{A}$ be any armature of A. Let $\mathcal{K} \subseteq \mathcal{A}$ be the kernel of the homomorphism $\bar{w}: \mathcal{A} \to \Delta/\Gamma_F$ of (3.5), and $\mathcal{K}^\perp$ the orthogonal dual of $\mathcal{K}$ in $\mathcal{A}$ with respect to the armature pairing $B_{\mathcal{A}}$. If D is the underlying division algebra of A, then $[D : F] = |\mathcal{K}^\perp : (\mathcal{K} \cap \mathcal{K}^\perp)|$, $\Gamma_D/\Gamma_F = \bar{w}(\mathcal{K}^\perp) \subseteq \Delta/\Gamma_F$, and the canonical pairing on Γ_D/Γ_F is isometric (via $\bar{w}$) to the pairing on $\mathcal{K}^\perp/(\mathcal{K} \cap \mathcal{K}^\perp)$ induced by $B_{\mathcal{A}}$. In particular,*
  (i) *A is a division algebra if and only if $\bar{w}$ is injective.*
  (ii) *A is split (i.e., $D = F$) if and only if $\mathcal{K} \supseteq \mathcal{K}^\perp$.*

PROOF. We have $A \cong M_t(D) \cong M_t(F) \otimes_F D$, for some t with $\mathrm{char}(\overline{F}) \nmid t$. Since $\mu_t \subseteq F$ and Draxl's Theorem (3.7) applies to D, A is a tensor product of symbol algebras; this assures the existence of an armature $\mathcal{A}$ for A. Say $\dim_F A = n^2$.

Consider first the extreme case in which $\mathcal{K}^\perp \subseteq \mathcal{K}$. Then $\mathcal{K}^\perp$ is totally isotropic, and there is a Lagrangian $\mathcal{L} \supseteq \mathcal{K}^\perp$. We have $|\mathcal{L}| = \sqrt{|\mathcal{A}|} = n$ and $\mathcal{K}^\perp \subseteq \mathcal{L} = \mathcal{L}^\perp \subseteq \mathcal{K}^{\perp\perp} = \mathcal{K}$. Let $L = F[\mathcal{L}]$, which is commutative, as $\mathcal{L}$ is totally isotropic. Take any $\tilde{c} = cF^* \in \mathcal{L}$, and let $k = \mathrm{ord}(\tilde{c})$. Then, as $\mathcal{L} \subseteq \mathcal{K}$, $v(c^k) \in k\Gamma_F$; because F is strictly Henselian and $\mathrm{char}(\overline{F}) \nmid k$, $c^k \in F^{*k}$. It follows by (2.9(b)) that $L \cong F \oplus \cdots \oplus F$ (n times). Since A contains a family of n orthogonal idempotents, by $[\mathbf{A_3}, \text{p. 50, Theorem 2}]$ A must be split, as desired.

Now consider the general case. Note that $B_{\mathcal{A}}$ restricts to a symplectic pairing on $\mathcal{K}^\perp$ whose radical is $\mathcal{K} \cap \mathcal{K}^\perp$. Hence $B_{\mathcal{A}}$ induces a nondegenerate symplectic pairing on $\mathcal{K}^\perp/(\mathcal{K} \cap \mathcal{K}^\perp)$. Let Γ be the inverse image in Δ of $\bar{w}(\mathcal{K}^\perp) \subseteq \Delta/\Gamma_F$, and let $\bar{w}': \mathcal{K}^\perp/(\mathcal{K} \cap \mathcal{K}^\perp) \to \Gamma/\Gamma_F$ be the isomorphism induced by $\bar{w}$. Let C be the nondegenerate symplectic pairing on Γ/Γ_F with values in $\mu(\overline{F})$ induced via $\bar{w}'$ by the pairing on $\mathcal{K}^\perp/(\mathcal{K} \cap \mathcal{K}^\perp)$. From the part of Proposition 4.2 already proved there is a tame F-central division algebra E with $\Gamma_E = \Gamma$ and C for its canonical pairing. Let $\mathcal{E}$ be any armature of E, and $\bar{v}: \mathcal{E} \to \Gamma_E/\Gamma_F$ the isomorphism of (3.3). Let $g: \mathcal{K}^\perp \to \mathcal{E}$ be the composition of surjective maps

$$\mathcal{K}^\perp \to \mathcal{K}^\perp/(\mathcal{K} \cap \mathcal{K}^\perp) \xrightarrow{\bar{w}'} \Gamma_E/\Gamma_F \xrightarrow{\bar{v}^{-1}} \mathcal{E}.$$

Then, for any $\tilde{p}, \tilde{r} \in \mathcal{K}^\perp$,

$$(*) \qquad \bar{v}(g(\tilde{p})) = \bar{w}(\tilde{p}) \quad \text{and} \quad B_{\mathcal{E}}(g(\tilde{p}), g(\tilde{r})) = B_{\mathcal{A}}(\tilde{p}, \tilde{r}).$$

We must show that $D \cong E$, where D is the underlying division algebra of A. For this it suffices to verify that $A \otimes_F E^{\mathrm{op}}$ is split, where E^{op} is the opposite algebra of E. The elements of $\mathcal{E}$ form an armature of E^{op}, which we denote $\mathcal{E}'$ because $\mathcal{E}$ and $\mathcal{E}'$ have different armature pairings: namely $B_{\mathcal{E}'}(\tilde{e}, \tilde{f}) = B_{\mathcal{E}}(\tilde{f}^{-1}, \tilde{e}^{-1}) = B_{\mathcal{E}}(\tilde{e}, \tilde{f})^{-1}$.

Let $A \perp \mathcal{E}' = \{a \widetilde{\otimes} e | \ \tilde{a} \in A \text{ and } \tilde{e} \in \mathcal{E}'\}$, which is an armature for $A \otimes_F E^{\mathrm{op}}$ (cf. (2.4(d)). For convenience, write $(\tilde{a}, \tilde{e})$ for the element $a \widetilde{\otimes} e$ of $A \perp \mathcal{E}'$. Let $\bar{y} \colon A \perp \mathcal{E}' \to \Delta/\Gamma_F$ be the map of (3.5), let $\mathcal{M} = \ker(\bar{y}) \subseteq A \perp \mathcal{E}'$, and let $\mathcal{M}^\perp$ be the orthogonal dual of $\mathcal{M}$ with respect to $B_{A\perp\mathcal{E}'}$. We will show that $\mathcal{M}^\perp \subseteq \mathcal{M}$; then the earlier case shows $A \otimes_F E^{\mathrm{op}}$ is split.

Take any $(\tilde{a}, \tilde{e}) \in \mathcal{M}^\perp$. Since g is surjective there is a $\tilde{p} \in K^\perp$ with $g(\tilde{p}) = \tilde{e}$. Now, take any $\tilde{r} \in K^\perp$. Then $\bar{y}(\tilde{r}, g(\tilde{r})^{-1}) = \bar{w}(\tilde{r}) + \bar{v}(g(\tilde{r})^{-1}) = 0$ by $(*)$ above. So, $(\tilde{r}, g(\tilde{r})^{-1}) \in \mathcal{M}$, and we have, using $(*)$,

$$\begin{aligned}
1 &= B_{A\perp\mathcal{E}'}((\tilde{a}, \tilde{e}), (\tilde{r}, g(\tilde{r})^{-1})) = B_A(\tilde{a}, \tilde{r}) \cdot B_{\mathcal{E}'}(\tilde{e}, g(\tilde{r})^{-1}) \\
&= B_A(\tilde{a}, \tilde{r}) \cdot B_{\mathcal{E}}(g(\tilde{p}), g(\tilde{r})) = B_A(\tilde{a}, \tilde{r}) \cdot B_A(\tilde{p}, \tilde{r}) \\
&= B_A(\tilde{a}\tilde{p}, \tilde{r}).
\end{aligned}$$

Since this is true for all $\tilde{r} \in K^\perp$ we have $\tilde{a}\tilde{p} \in K^{\perp\perp} = K$. Thus,

$$0 = \bar{w}(\tilde{a}\tilde{p}) = \bar{w}(\tilde{a}) + \bar{w}(\tilde{p}) = \bar{w}(\tilde{a}) + \bar{v}(\tilde{e}) = \bar{y}(\tilde{a}, \tilde{e}).$$

Hence $(\tilde{a}, \tilde{e}) \in \mathcal{M}$, which shows $\mathcal{M}^\perp \subseteq \mathcal{M}$. Therefore, $D \cong E$ as desired, and statements (i) and (ii) follow at once. (Of course, (i) was already known from (3.3) and (3.5).)   □

PROOF OF PROPOSITION 4.2, COMPLETED. Let D_1 and D_2 be two tame F-central division algebras with the same value group and the same canonical pairing. Then for any armature $\mathcal{D}_1$ of D_1 we have by (3.3) and (3.4) that $\bar{v} \colon \mathcal{D}_1 \to \Gamma_{D_1}/\Gamma_F$ is an isomorphism and an isometry between $B_{\mathcal{D}_1}$ and the canonical pairing of D_1. Then, the proof of Theorem 4.3 with $A = D_1$ and $E = D_2$ shows that D_2 is the underlying division algebra of D_1, i.e., $D_2 \cong D_1$. This gives the uniqueness part of (4.2).

Now take any two tame F-central division algebras D and D'; suppose $\Gamma_{D'} \subseteq \Gamma_D$ and the canonical pairing of D restricts to the canonical pairing of D'. Let $\mathcal{D}$ be any armature of D, and $\bar{v} \colon \mathcal{D} \to \Gamma_D/\Gamma_F$ the isomorphism of (3.3). Let $\mathcal{D}' = \bar{v}^{-1}(\Gamma_{D'}/\Gamma_F) \subseteq \mathcal{D}$. Then $F[\mathcal{D}']$ and D' have the same value group and canonical pairing. By what we just proved, $D' \cong F[\mathcal{D}'] \subseteq D$, as desired. Since the converse part of the last statement of (4.2) is trivial, (4.2) is now fully proved.   □

Here are some further applications of the theorem. We continue to assume that (F, v) is strictly Henselian.

EXAMPLES 4.4. (i) Let D_1 and D_2 be two F-central division algebras, with value groups Γ_1 and Γ_2 and canonical pairings C_1 and C_2. We determine the underlying division algebra D of $D_1 \otimes_F D_2$. Let $\Phi = (\Gamma_1/\Gamma_F) \times (\Gamma_2/\Gamma_F)$. Then $C_1 \perp C_2$ is a nondegenerate symplectic pairing on Φ, and there is a homomorphism $f \colon \Phi \to \Delta/\Gamma_F$ given by $(\bar{\alpha}, \bar{\beta}) \mapsto \bar{\alpha} + \bar{\beta}$. Let $K = \ker(f) \subseteq \Phi$ and $K^\perp$ the orthogonal dual of K in Φ with respect to $C_1 \perp C_2$. Then $[D : F] = |K^\perp : (K \cap K^\perp)|$, $\Gamma_D/\Gamma_F = f(K^\perp)$, and the canonical pairing on Γ_D/Γ_F is isometric (via f) to the pairing on $K^\perp/(K \cap K^\perp)$ induced by $C_1 \perp C_2$ on Φ. This is easily verified from the theorem using the isometry between the canonical pairings C_i and the armature pairings $B_{\mathcal{D}_i}$ for any armature $\mathcal{D}_i$ of D_i, $i = 1, 2$, together with Example 2.4(d) above.

(ii) Consider now a tame F-central division algebra D with canonical pairing C_D, and let D' be the underlying division algebra of $D \otimes_F D \otimes_F \cdots \otimes_F D$, the mth tensor power of D, for any natural number m. By the same approach as for

(i) (or by repeated applications of (i)), one can see that D' is the tame F-division algebra with $[D' : F] = |(m\Gamma_D + \Gamma_F) : \Gamma_F|$, $\Gamma_{D'} = m\Gamma_D + \Gamma_F$, and the canonical pairing on $\Gamma_{D'}$ is given by $C_{D'}(m\overline{\alpha}, m\overline{\beta}) = C_D(\overline{\alpha}, \overline{\beta})^m$ for all $\overline{\alpha}, \overline{\beta} \in \Gamma_D/\Gamma_F$. Thus, the order of $[D]$ in the Brauer group $\mathrm{Br}(F)$ is clearly $\exp(\Gamma_D/\Gamma_F)$. (This formula for the order of $[D]$ can also be verified easily using the decomposition of D into symbol algebras.)

The theorem also allows us to determine what happens to D upon extension of scalars:

PROPOSITION 4.5.   *Let (F, v) be a strictly Henselian valued field, and D a tame F-central division algebra with canonical pairing C_D. Let Δ be the divisible hull of Γ_F. Let E be any finite degree field extension of F, and let D' be the underlying division algebra of $D \otimes_F E$. Let $\Psi = (\Gamma_D \cap \Gamma_E)/\Gamma_F \subseteq \Delta/\Gamma_F$, and $\Psi^\perp$ the orthogonal dual of Ψ in Γ_D/Γ_F with respect to C_D. Then $[D' : E] = |\Psi^\perp : (\Psi \cap \Psi^\perp)|$, $\Gamma_{D'}/\Gamma_E$ is the image of $\Psi^\perp$ in $(\Gamma_D + \Gamma_E)/\Gamma_E$, and the canonical pairing on $\Gamma_{D'}/\Gamma_E$ is isometric to the pairing on $\Psi^\perp/(\Psi \cap \Psi^\perp)$ induced by C_D. In particular,*

(i) *E splits D if and only if $\Psi \supseteq \Psi^\perp$.*
(ii) *$D \otimes_F E$ is a division algebra if and only if $\Gamma_E \cap \Gamma_D = \Gamma_F$.*

PROOF. Note that E is strictly Henselian with respect to the unique extension of v to E, and $D \otimes_F E$ is a tame E-algebra. Let $\mathcal{D}$ be any armature of D, and $\overline{v}: \mathcal{D} \to \Gamma_D/\Gamma_F$ the isomorphism induced by v. Now, $\mathcal{D}_E := \{\widetilde{d \otimes 1} | \ \widetilde{d} \in \mathcal{D}\} \subseteq (D \otimes_F E)^*/E^*$ is an armature of $D \otimes_F E$ as an E-algebra (cf. (2.4)(e)), and the natural map $f: \mathcal{D} \to \mathcal{D}_E$ is an isomorphism which is also an isometry between the armature pairings $B_\mathcal{D}$ and $B_{\mathcal{D}_E}$. Let $g: \Delta/\Gamma_F \to \Delta/\Gamma_E$ be the natural projection, let $\overline{w}: \mathcal{D}_E \to \Delta/\Gamma_E$ be the map of (3.5), and let $K = \ker(\overline{w})$. Since $\overline{w} \circ f = g \circ \overline{v}$, we have $\overline{v} \circ f^{-1}(K) = \ker(g) \cap (\Gamma_D/\Gamma_F) = \Psi$. Note that $\overline{v} \circ f^{-1}$ is an isometry between $B_{\mathcal{D}_E}$ on $\mathcal{D}_E$ and C_D on Γ_D/Γ_F. Hence, $\overline{v} \circ f^{-1}(K^\perp) = \Psi^\perp$, and the proposition follows immediately from Theorem 4.3. $\square$

This proposition yields an easy nonhomological proof of the main result of [**TA$_1$**]:

COROLLARY 4.6 (TIGNOL, AMITSUR).   *Let $F, v, D,$ and E be as in Proposition 4.5. Then E splits D if and only if E contains an isomorphic copy of a maximal subfield of D.*

PROOF. Suppose E splits D. Let E_0 be the maximal tame extension of F in E. Then E_0 splits D since $[E : E_0]$ is a power of $\mathrm{char}(\overline{F})$ which is prime to $[D : F]$. Let $\Psi = (\Gamma_{E_0} \cap \Gamma_D)/\Gamma_F$. By Proposition 4.5, $\Psi \supseteq \Psi^\perp$. Hence, $\Psi^\perp$ is totally isotropic with respect to the pairing C_D on Γ_D/Γ_F, and we can find a Lagrangian Λ of Γ_D/Γ_F with $\Psi^\perp \subseteq \Lambda$. Then $\Psi^\perp \subseteq \Lambda = \Lambda^\perp \subseteq \Psi$. Let L be the subalgebra of D associated to Λ in the correspondence of (3.8). Then $\Gamma_L/\Gamma_F = \Lambda$ and, since Λ is a Lagrangian of Γ_D/Γ_F, L is a maximal subfield of D. Since $\Gamma_L/\Gamma_F \subseteq \Psi \subseteq \Gamma_{E_0}/\Gamma_F$, Proposition 4.1 shows that L is isomorphic to a subfield of E_0, hence of E, as desired. The converse is clear. $\square$

Using the extension Lemma 1.6 we can carry over some of the information obtained in the strictly Henselian case to totally ramified division algebras over an arbitrary field.

THEOREM 4.7.   *Let (D, v) be a valued division algebra which is tame and totally ramified over its center F, which can be any valued field. Let C_D be the canonical pairing on Γ_D/Γ_F. Then,*

(i) *$\exp(\Gamma_D/\Gamma_F)$ divides the order of $[D]$ in $\mathrm{Br}(F)$.*

(ii) *Let $K \supseteq F$ be any field, and let D' be the underlying division algebra of $D \otimes_F K$. If w is any extension of v to K, then $[D' : K]$ is a multiple of $|\Psi^\perp : (\Psi \cap \Psi^\perp)|$, where $\Psi = (\Gamma_K \cap \Gamma_D)/\Gamma_F$ and $\Psi^\perp \subseteq \Gamma_D/\Gamma_F$ is taken relative to C_D. Equality holds if w extends to a valuation on D' totally ramified over K.*

(iii) *For (K, w) as in (ii) with $[K : F] < \infty$ suppose K splits D. Then*

$$\mathrm{rank}(\Gamma_K/\Gamma_F) \geq \tfrac{1}{2}\,\mathrm{rank}(\Gamma_D/\Gamma_F).$$

If K is Galois over F, then $\mathcal{G}(K/F)$ has a subgroup isomorphic to a Lagrangian of Γ_D/Γ_F.

(iv) *If*

$$[D] = \left[\bigotimes_{i=1}^{k}{}_F A_{\omega_i}(b_i, c_i; F)\right]$$

in the Brauer group $\mathrm{Br}(F)$, then $k \geq \tfrac{1}{2}\,\mathrm{rank}(\Gamma_D/\Gamma_F)$.

(v) *In particular, if $\mathrm{rank}(\Gamma_D/\Gamma_F) > 2$, then D is not split by any cyclic Galois extension of F.*

PROOF. Let F_s be a separable closure of F, and v_s any extension of v to F_s. Let F' be the inertia field of v_s over v (cf. [**E**, §19]), and $v' = v_s|_{F'}$. Then (F', v') is strictly Henselian and $\Gamma_{F'} = \Gamma_F$. By the extension lemma 1.6, $D_1 := D \otimes_F F'$ is a division ring with $\Gamma_{D_1} = \Gamma_D$. Thus, using 4.4(ii), we have $\exp(\Gamma_D/\Gamma_F) = \exp(\Gamma_{D_1}/\Gamma_{F'}) = \mathrm{ord}[D_1]$. Since $\mathrm{ord}[D_1]\,|\,\mathrm{ord}[D]$, this yields (i).

Now, given (K, w) as in (ii), construct (K_s, w_s) and (K', w') in the same manner as we just constructed (F_s, v_s) and (F', v') from (F, v). We may assume that $F_s \subseteq K_s$ and $v_s = w_s|_{F_s}$; so $F' \subseteq K'$. Let D_2 be the underlying division algebra of $D' \otimes_K K'$ (which is also the underlying division algebra of $D_1 \otimes_{F'} K'$). Since $\Psi = (\Gamma_K \cap \Gamma_D)/\Gamma_F = (\Gamma_{K'} \cap \Gamma_{D_1})/\Gamma_{F'}$, we have

$$|\Psi^\perp : (\Psi \cap \Psi^\perp)| = [D_2 : K']$$

(which divides $[D : K']$) by the argument for Proposition 4.5 applied to K' over F'. (The proof of (4.5) uses E and F strictly Henselian, but does not require $[E : F] < \infty$.) If w extends to a valuation on D' totally ramified over K, then the extension lemma shows $D_2 = D' \otimes_K K'$, hence $[D' : K] = [D_2 : K']$, completing the proof of (ii).

For (iii), suppose K splits D. Then (ii) shows $\Psi^\perp \subseteq \Psi$. Hence there is a Lagrangian Λ of Γ_D/Γ_F with $\Lambda \subseteq \Psi \subseteq \Gamma_K/\Gamma_F$, which yields $\mathrm{rank}(\Gamma_K/\Gamma_F) \geq \mathrm{rank}(\Lambda) \geq \tfrac{1}{2}\,\mathrm{rank}(\Gamma_D/\Gamma_F)$ (cf. (2.3)). If K is Galois over F, then $K_s = F_s$ and $K' = F' \cdot K$. Hence K' is Galois over F' and $\mathcal{G}(K'/F')$ injects into $\mathcal{G}(K/F)$. Let M be the ramification field of w' over v'. Then $F' \subseteq M \subseteq K'$, M is Galois over F', and $\mathcal{G}(K'/M)$ is the unique p-Sylow subgroup of $\mathcal{G}(K'/F')$ where $p = \mathrm{char}(\overline{F})$ (cf. [**E**, (20.11)]). Because K' splits D and $[K' : M]$ is prime to $[D : F]$, D must already be split by M. The argument just given shows Γ_M/Γ_F contains a Lagrangian Λ of Γ_D/Γ_F. But $\mathcal{G}(M/F') \cong \Gamma_M/\Gamma_{F'}$ as M is tame over F' which is strictly Henselian. Furthermore, by the Schur-Zassenhaus lemma [**Ro**, p. 151] since

$\mathcal{G}(K'/M)$ is normal in $\mathcal{G}(K'/F')$ and since $\mathcal{G}(K'/M)$ and $\mathcal{G}(M/F')$ have relatively prime orders, $\mathcal{G}(M/F')$ is isomorphic to a subgroup of $\mathcal{G}(K'/F')$. Thus we have a succession of injective maps $\Lambda \hookrightarrow \Gamma_M/\Gamma_{F'} \hookrightarrow \mathcal{G}(M/F') \hookrightarrow \mathcal{G}(K'/F') \hookrightarrow \mathcal{G}(K/F)$, proving (iii).

Now suppose $[D] = [\bigotimes_{i=1}^k A_{\omega_i}(b_i, c_i; F)]$ in $\mathrm{Br}(F)$. Let $n_i = \mathrm{ord}(\omega_i)$ in F^*, and let K be the field $F(b_1^{1/n_1}, \ldots, b_k^{1/n_k})$. Then K is a Kummer extension of F, and K splits D since it splits each of the symbol algebras. Because $\mathcal{G}(K/F)$ contains a Lagrangian Λ of Γ_D/Γ_F by (iii), we have $k \geq \mathrm{rank}(\mathcal{G}(K/F)) \geq \mathrm{rank}(\Lambda) \geq \frac{1}{2}\mathrm{rank}(\Gamma_D/\Gamma_F)$, proving (iv).

Part (v) is immediate from (iii) since if $\mathrm{rank}(\Gamma_D/\Gamma_F) > 2$, Γ_D/Γ_F has no cyclic Lagrangians. $\square$

Parts (iii) and (iv) of Theorem 4.7 are generalizations of [**TA₂**, Theorem 4.4] and the proof of [**T₂**, Lemma 4.1].

REMARK 4.11. The Brauer group $\mathrm{Br}(F)$ of a strictly Henselian field F is known (except for the $\mathrm{char}(\overline{F})$-primary component), and has a nice homological description (cf. [**Sch**, or **W₁**]). Bill Jacob suggested to us that there ought also to be a homological interpretation of the process for finding the underlying division algebra in Theorem 4.3. From this perspective the theorem turns out to be equivalent to a result on exterior products of **Z**-modules. We now sketch this approach.

Let (F, v) be a strictly Henselian valued field, and let p be a prime number, $p \neq \mathrm{char}(\overline{F})$. So, $\mu_{p^n} \subseteq F$ for all n. Let $\tilde{F}^p$ be the p-closure of F, which is the compositum of all finite-degree Galois extensions K of F with $[K : F]$ a power of p. Then $\tilde{F}^p$ is a Galois extension of F with pro-p Galois group $G := \mathcal{G}(\tilde{F}_p/F)$. Because F is strictly Henselian G is a *free abelian* pro-p group. Also, since $\tilde{F}_p$ has no field extension of p-power degree the relative Brauer group $\mathrm{Br}(\tilde{F}_p/F)$ ($\cong H^2(G, \tilde{F}_p^*)$) is the full p-primary component of $\mathrm{Br}(F)$. Hence, for each n the continuous cohomology group $H^2(G, \mu_{p^n})$ is the full p^n-torsion subgroup of $\mathrm{Br}(F)$.

Because G is free abelian, the cup product pairing

$$\cup \colon H^1(G, \mu_{p^n}) \times H^1(G, \mu_{p^n}) \to H^2(G, \mu_{p^n})$$

induces an isomorphism

$$H^1(G, \mu_{p^n}) \wedge H^1(G, \mu_{p^n}) \cong H^2(G, \mu_{p^n}),$$

where $\wedge$ denotes the exterior product (as **Z**-modules). Of course, as $\mu_{p^n} \subseteq F$,

$$H^1(G, \mu_{p^n}) \cong F^*/F^{*p^n} \cong \Gamma_F/p^n\Gamma_F.$$

Finding the underlying division algebra of a $\gamma \in H^2(G, \mu_{p^n})$ corresponds to finding a minimal representation of γ in the form $\gamma = \sum_{i=1}^k \alpha_i \wedge \beta_i$ with $\alpha_i, \beta_i \in H^1(G, \mu_{p^n})$ (minimal in the sense that $\sum_{i=1}^k \mathrm{ord}(\alpha_i \wedge \beta_i)$ is minimal). The method for finding D in Theorem 4.3 generalizes to give an algorithm for finding a minimal representation of any element of $M \wedge M$ (exterior product as R/I-modules) where M is any R/I-module, for R a principal ideal domain (or Dedekind domain) and I any nonzero ideal of R. As we describe this more explicitly, we restrict for simplicity to the case where R/I is a field.

Suppose, then, that M is a vector space of any dimension over any field F. Take any $\gamma \in M \wedge M$. A minimal representation of γ is an expression $\gamma = \sum_{i=1}^k \alpha_i \wedge \beta_i$

with k minimal. The representation is easily seen to be minimal if and only if $\mathcal{B} := \{\alpha_1, \beta_1, \ldots, \alpha_k, \beta_k\}$ is linearly independent over F. When this linear independence holds, let $V_{\mathcal{B}} = \text{span}\,\mathcal{B}$, and let $B_{\mathcal{B}} : V \times V \to F$ be the nondegenerate symplectic F-bilinear pairing on V with symplectic base $\mathcal{B}$. It is presumably well-known that $\sum_{i=1}^{k} \alpha_i' \wedge \beta_i'$ is another minimal representation of γ if and only if $\{\alpha_i', \beta_i', \ldots, \alpha_k', \beta_k'\}$ is another symplectic base of $B_{\mathcal{B}}$ on $V_{\mathcal{B}}$. Thus, the elements of $M \wedge M$ are in 1-1 correspondence with pairs (V, B), where V is an even (finite) dimensional subspace of M and B is a nondegenerate symplectic pairing on V.

Now, suppose we have a (possibly nonminimal) representation $\gamma = \sum_{i=1}^{l} \delta_i \wedge \varepsilon_i$ of $\gamma \in M \wedge M$. The method for finding (V, B) corresponding to γ analogous to Theorem 4.3 runs as follows: Let $\mathcal{C} = \{x_1, y_1, \ldots, x_l, y_l\}$ be a base of a vector space W, let B_W be the symplectic pairing on W with $\mathcal{C}$ as a symplectic base, and let $f : W \to M$ be the F-linear transformation given by $x_i \mapsto \delta_i$ and $y_i \mapsto \varepsilon_i$, $1 \le i \le l$. Let $K = \ker(f)$ and $K^{\perp}$ be the orthogonal dual of K with respect to B_W. Then $V = f(K^{\perp})$ and B is the symplectic pairing on V isometric (via f) to the one induced on $K^{\perp}/(K \cap K^{\perp})$ by B_W. The proof is omitted.

5. Examples. As applications of the theory developed in the preceding sections, we now construct two examples.

The first example is a pair of division algebras D_1, D_2 central simple over a field F such that $D_1 \otimes_F D_2$ is not a division algebra but D_1 and D_2 have no common subfield properly containing F. This answers an old question dating back to Albert's work in $[\mathbf{A_1}]$. Albert showed that if Q_1 and Q_2 are quaternion algebras over F and $Q_1 \otimes Q_2$ is not a division algebra, then Q_1 and Q_2 have a common maximal subfield. Of course, it was not really expected that Albert's result would extend to arbitrary finite-dimensional division algebras. The main obstacle in producing a counterexample has been the difficulty in determining all the subfields of a given division algebra.

For this example, let n be any *odd* integer, $n > 1$, and let k be any field containing a primitive nth root of unity ω. Let $F = k(t, x, y)$, where t, x, y are independent indeterminates over k. Let

$$D_1 := A_{\omega}(x, y; F) \quad \text{and} \quad D_2 := A_{\omega}(x(t-1)/y, xt; F).$$

PROPOSITION 5.1. *D_1 and D_2 are division algebras, $D_1 \otimes_F D_2$ is not a division algebra, but there is no field $K \supsetneq F$ isomorphic to a subfield of D_1 and also to a subfield of D_2.*

PROOF. Let $z = x(t-1)/y$ and $\pi = xt = x + yz$. In $D_1 \otimes_F D_2$ we have elements i_1, j_1, i_2, j_2 which commute pairwise except that $i_1 j_1 = \omega j_1 i_1$ and $i_2 j_2 = \omega j_2 i_2$; further $i_1^n = x$, $j_1^n = y$, $i_2^n = z$, $j_2^n = \pi$. Let $\alpha = j_1 i_1^{-1} i_2$, $\gamma = i_1 j_2^{-1}$, and $\beta = (1 + \alpha)\gamma = (i_1 + j_1 i_2)j_2^{-1}$. Since $\gamma \alpha \gamma^{-1} = \omega^2 \alpha$, we have

$$\beta^n = (1 + \alpha)\gamma(1 + \alpha)\gamma^{-1}\gamma^2(1 + \alpha)\gamma^{-2} \cdots \gamma^{n-1}(1 + \alpha)\gamma^{-(n-1)}\gamma^n$$
$$= (1 + \alpha)(1 + \omega^2 \alpha) \cdots (1 + \omega^{2(n-1)}\alpha)\gamma^n.$$

Recall the cyclotomic identity $X^n - 1 = \prod_{i=0}^{n-1}(X - \rho^i)$, for any primitive nth root of unity ρ. Since n is odd, this yields $X^n + 1 = -[(-X)^n - 1] = \prod_{i=0}^{n-1}(X + \rho^i)$.

Since $\rho = \omega^{-2}$ is a primitive nth root of unity, we have

$$\prod_{i=0}^{n-1} (1 + \omega^{2i}\alpha) = (\omega^2)^{n(n-1)/2} \prod_{i=0}^{n-1} (\omega^{-2i} + \alpha) = 1 \cdot (1 + \alpha^n).$$

Therefore,

$$\beta^n = (1 + \alpha^n)(i_1 j_2^{-1})^n = (1 + yx^{-1}z)(x\pi^{-1}) = (x + yz)\pi^{-1} = 1.$$

Since β is not central in $D_1 \otimes_F D_2$, $\beta \notin F$. Thus, $F(\beta)$ cannot be a field, since it contains $n + 1$ different nth roots of unity. Therefore, $D_1 \otimes_F D_2$ is not a division ring.

(Alternatively, one can check, using the identities for symbol algebras, that in the Brauer group $\mathrm{Br}(F)$

$$[D_1 \otimes_F D_2] = [A_\omega(x, y; F)] + [A_\omega(x(t-1)/y, xt; F)]$$
$$= [A_\omega(x, ty^2/(t-1); F)] + [A_\omega(t, y; F)] = [A_\omega(xt^s, ty^2/(t-1); F)],$$

where $s = (n+1)/2$, an integer.)

It remains to see that D_1 and D_2 are division algebras with "no" common subfields. For this, let $F' = k(t)((x))((y))$, the twice iterated Laurent power series field over $k(t)$, and let $D_i' = D_i \otimes_F F'$. It suffices to see that D_1' and D_2' are division algebras with "no" common subfields. For, if $K \supsetneq F$ is a field embedding in D_1 and D_2 then $K \otimes_F F'$ embeds in D_1' and D_2'. The embedding in the D_i' assures that $K \otimes_F F'$ is a field.

Let v be the standard Henselian valuation on F', as described in (3.6) above. So, $\overline{F'} = k(t), \Gamma_{F'} = \mathbf{Z} \times \mathbf{Z}$, and $v(x) = (1, 0)$, $v(y) = (0, 1)$. Because $v(x), v(y)$ map onto a base of $\Gamma_F/n\Gamma_F$, (3.5) (or [**JW**, Corollary 2.6]) shows that $D_1' = A_\omega(x, y; F')$ is a division algebra and the extension of v to D_1' is totally ramified over F', with $\Gamma_{D_1'} = (1/n)\mathbf{Z} \times (1/n)\mathbf{Z}$. By Theorem 3.8 the subfields L of $D_1', L \supsetneq F'$ are classified by the nonidentity totally isotropic subgroups of $\Gamma_{D_1'}/\Gamma_{F'}$. The minimal proper field extensions of F' in D_1' correspond to minimal nonidentity subgroups of $\Gamma_{D_1'}/\Gamma_{F'}$. Every such minimal subgroup is generated by $(i/p, j/p) \pmod{\mathbf{Z} \times \mathbf{Z}}$ for some prime p dividing n and integers i, j, not both mutiples of p; the corresponding field must be $F(\sqrt[p]{x^i y^j})$ since this is a subfield of D_1' with the right value group.

Now, $D_2' = A_\omega(z, \pi; F')$ where $z = x(t-1)/y$ and $\pi = xt$. Since $v(z) = (1, -1)$ and $v(\pi) = (1, 0)$, the same analysis for D_1' shows D_2' is a division algebra, the extension of v from F' to D_2' is totally ramified with $\Gamma_{D_2'} = (1/n)\mathbf{Z} \times (1/n)\mathbf{Z}$, and every subfield L of $D_2', L \supsetneq F'$ contains a subfield of the form $F'(\sqrt[p]{z^l \pi^m})$ with $p|n$ prime and $l, m \in \mathbf{Z}$, not both multiples of p. Thus, if D_1' and D_2' have a common subfield we must have $F'(\sqrt[p]{x^i y^j}) = F'(\sqrt[p]{z^l \pi^m}) \supsetneq F'$ for suitable p, i, j, l, m. By Kummer theory $z^l \pi^m \equiv (x^i y^j)^q \pmod{F'^{*p}}$, for some integer q. By comparing the values of these expressions we find $(iq, jq) \equiv (l + m, -l) \pmod{p\mathbf{Z} \times p\mathbf{Z}}$, i.e., $iq \equiv (l + m) \pmod{p}$ and $jq \equiv -l \pmod{p}$. Hence F'^{*p} contains $z^l \pi^m x^{-(l+m)} y^l = (t-1)^l t^m$. This unit must then map to a pth power in the residue field $\overline{F'} = k(t)$. Hence, $p|l$ and $p|m$, so that $F'(\sqrt[p]{z^l \pi^m}) = F'$, a contradiction. $\square$

REMARK. David Saltman has recently communicated the following observation about this example: Although $D_1 \otimes_F D_2$ is not a division algebra, $D_1^{\mathrm{op}} \otimes_F D_2$ *is* a division algebra. (For, by the identities for symbol algebras, $D_1^{\mathrm{op}} \otimes_F D_2 \cong$

$A_\omega((t-1)/t, x; F) \otimes_F A_\omega(t, y; F)$. By [**JW**, Corollary 2.9] the latter tensor product is a division algebra since it has a valuation (whose restriction to F coincides with the valuation F inherits from F').) As D_1 and D_1^{op} have the same subfields, there can be "no" common subfield to D_1 and D_2. Saltman raises the following question: If D_1 and D_2 are any F-central division algebras of prime index p and if $D_1 \otimes_F D_2^{(i)}$ is not a division algebra for $1 \le i \le p-1$ (where $D_2^{(i)}$ is the underlying division algebra of the ith tensor power of D_2), must D_1 and D_2 have a common subfield? Albert's theorem says yes if $p = 2$. But for $p \ge 3$ the question is open.

Our second example is a division algebra D which is not expressible as a tensor product of symbol algebras. The first such algebra was constructed by Albert in [**A$_2$**]. Also, it is known that the generic division algebra $UD(F, n)$ is not a tensor product of symbol algebras if $p^2 | n$ for some prime p. ($UD(F, n)$ is the quotient division algebra of the F-algebra generated by at least two generic $n \times n$ matrices over F.) What is new here is the method of construction: We will arrange to have different valuations on D, so that no possible tensor decomposition of D is compatible with all the relative value groups. The approach of playing off one valuation against another seems very useful for building counterexamples. This method was employed in the construction of noncrossed products in [**JW**].

Let r and s be any integers, $r > 1, s > 1$. We begin with fields $F \subseteq K$, with $\mu_{rs} \subseteq F$ and K Galois over F with abelian Galois group with base $\{\rho, \sigma\}$ where $\mathrm{ord}(\rho) = r$ and $\mathrm{ord}(\sigma) = s$. (Further conditions on F and K will be imposed below.) Let L_1 be the fixed field of ρ and L_2 the fixed field of σ. So, $[K : L_1] = [L_2 : F] = r$ and $[K : L_2] = [L_1 : F] = s$. Let $h \in K^*$ be any element with $N_{K/F}(h) = 1$, where $N_{K/F}$ denotes the norm from K to F. Then, let $D = K((i))((j))$, the iterated twisted Laurent series division algebra, where the multiplication on D is specified by the relations

(5.2) $ic = \rho(c)i, \quad jc = \sigma(c)j \quad$ for all $c \in K$, and $ij = hji$,

that is, D is the iterated twisted Laurent series division algebra

(5.3) $$D = K((i; \rho))((j; \sigma^*)),$$

in the notation of [**J$_2$**, p. 87], where σ^* is the extension of σ to $K((i; \rho))$ given by $\sigma^*(i) = h^{-1}i$. Clearly $K((i; \rho))$ has center $L_1((i^r))$ and $[K((i; \rho)) : L_1((i^r))] = r^2$. We have $\sigma^*(i^r) = (h^{-1}i)^r = N_{K/L_1}(h^{-1})i^r$. Since $N_{L_1/F}(N_{K/L_1}(h^{-1})) = N_{K/F}(h^{-1}) = 1$, by Hilbert 90 there is an $a \in L_1^*$ with $\sigma(a)/a = N_{K/L_1}(h^{-1})$. Then, as an automorphism of $L_1((i^r))$ σ^* has fixed field $F((a^{-1}i^r))$ and order s. Note that $\sigma^{*s}(i) = N_{K/L_2}(h^{-1})i$. Again by Hilbert 90 there is a $b \in L_2^*$ with $N_{K/L_2}(h^{-1}) = b/\rho(b)$. Then σ^{*s} agrees with conjugation by b on $K((i; \rho))$. Consequently, by [**J$_2$**, Theorem 5, p. 88], the center of D is $E := F((a^{-1}i^r))((b^{-1}j^s))$, and $[D : E] = (rs)^2$.

(Note: There are other ways of viewing D and E. One can check that D is a Malcev-Neumann algebra, as discussed in [**T$_1$**, §2] and [**TA$_2$**]. Also, one could start from the field $E = F((x))((y))$, extend ρ and σ to $K((x))((y))$ by $\rho(x) = \sigma(x) = x$ and $\rho(y) = \sigma(y) = y$, and define D to be the E-algebra generated over $K((x))((y))$ by i and j subject to the relations

$$ic = \rho(c)i, \quad jc = \sigma(c)j \quad \text{for all } c \in K((x))((y)),$$
$$i^r = ax, \quad j^s = by, \quad \text{and} \quad ij = hji.$$

From this description it is apparent that D is a power series version of the generic abelian crossed product of Amitsur and Saltman [**AS**, §2].)

There is a standard valuation $v \colon D^* \to \mathbf{Z} \times \mathbf{Z}$ on D given by

$$v\left(\sum_m \sum_n c_{m,n} i^m j^n\right) = \inf\{(m,n)|\, c_{m,n} \neq 0\}, \quad \text{for any } c_{m,n} \in K,$$

where $\mathbf{Z} \times \mathbf{Z}$ has the right-to-left lexicographical order. This is analogous to (3.6) for fields. It is easy to check that v is still a valuation, even though the multiplication on D is twisted by the relations (5.2). For this valuation, we have $\Gamma_{D,v} = \mathbf{Z} \times \mathbf{Z}$, $\overline{D}_v = K$, and $v(i) = (1,0)$, $v(j) = (0,1)$. The restriction of v to E is Henselian with $\Gamma_{E,v} = r\mathbf{Z} \times s\mathbf{Z}$ and $\overline{E}_v = F$.

Now, suppose F has a valuation w with a (unique) totally ramified extension to K, with $\mathrm{char}(\overline{F}_w) \nmid rs$. Then, as $\mu_{rs} \subseteq F$, $\overline{F}_w$ contains a primitive rs root of unity ε. Also, since $N_{K/F}(h) = 1$ and K/F is totally ramified, we must have $w(h) = 0$ and, in $\overline{K}_w$, $\bar{h}^{rs} = \overline{N_{K/F}(h)} = 1$. Hence,

$$(5.4) \qquad\qquad \text{in } \overline{K}_w, \quad \bar{h} = \varepsilon^t \quad \text{for some } t,\ 1 \leq t \leq rs.$$

Because w extends uniquely from F to K, every F-automorphism of K preserves w. Therefore, if $\pi \colon V_{D,v} \to \overline{D}_v$ is the natural projection, $\pi^{-1}(V_{K,w})$ is mapped to itself by all inner automorphisms of D. Since, in addition, $\pi^{-1}(V_{K,w})$ contains a or a^{-1} for each $a \in D^*$, $\pi^{-1}(V_{K,w})$ is a valuation ring of D (cf. [**S**, p. 17, Theorem 6]). Let u be the corresponding valuation on D. (Explicitly, $u(\sum_m \sum_n c_{m,n} i^m j^n) = \inf\{(w(c_{m,n}), (m,n))|\, c_{m,n} \neq 0\} \in \Gamma_{K,w} \times \Gamma_{D,v}$, where this product of value groups is ordered lexicographically right-to-left.) The usual exact sequence of value groups for composing valuations is valid here:

$$(5.5) \qquad\qquad 0 \to \Gamma_{K,w} \to \Gamma_{D,u} \to \Gamma_{D,v} \to 0.$$

LEMMA 5.6. *With respect to the valuation u, D is totally ramified over E. Further,*

$$\Gamma_{D,u}/\Gamma_{E,u} \cong (\mathbf{Z}/l\mathbf{Z})^2 \times (\mathbf{Z}/d\mathbf{Z})^2,$$

where $l = \mathrm{lcm}(rs, rt, st)/t$ (with t as in (5.4)) and $d = rs/l$.

PROOF. u restricts to a valuation on the field $E \cdot K = K((a^{-1}i^r))((b^{-1}j^s))$ and to one on E; in each case u is the composite valuation of the restrictions of v and w. Consider the exact sequence

$$(5.7) \qquad 0 \to \Gamma_{E \cdot K, u}/\Gamma_{E,u} \to \Gamma_{D,u}/\Gamma_{E,u} \to \Gamma_{D,u}/\Gamma_{E \cdot K, u} \to 0.$$

Since $\Gamma_{E \cdot K, v} = \Gamma_{E,v}$, we have $\Gamma_{E \cdot K, u}/\Gamma_{E,u} \cong \Gamma_{K,w}/\Gamma_{F,w}$ (as one can see by comparing the exact sequences (5.5) for $E \cdot K$ and E); likewise, since $\overline{D}_v = K = \overline{E \cdot K}_v$, we have $\Gamma_{D,u}/\Gamma_{E \cdot K, u} \cong \Gamma_{D,v}/\Gamma_{E \cdot K, v}$. Thus, with respect to u, D is totally ramified over $E \cdot K$ and $E \cdot K$ is totally ramified over E; hence D is totally ramified over E. Let $G = \Gamma_{D,u}/\Gamma_{E,u}$. Writing $\bar{u}$ for the composite map $D^* \to \Gamma_{D,u} \to \Gamma_{D,u}/\Gamma_{E,u}$, we see from (5.7) that G is generated by $\bar{u}(i), \bar{u}(j)$, and the subgroup $H := \Gamma_{E \cdot K, u}/\Gamma_{E,u}$. Note also that $r\bar{u}(i), s\bar{u}(j) \in H$ as $i^r, j^s \in E \cdot K$. The canonical pairing C_D for u on D is completely determined by the conditions:

$$C_D(\bar{u}(i), \bar{u}(c)) = \overline{\rho(c)/c}, \quad C_D(\bar{u}(j), \bar{u}(c)) = \overline{\sigma(c)/c} \quad \text{for all } c \in K^*,$$

$$C_D(\bar{u}(i), \bar{u}(j)) = \bar{h} = \varepsilon^t,$$

H is a Lagrangian (as $E \cdot K$ is a field).

Now, ε^s is a primitive rth root of unity. By Kummer theory (or by invoking (1.4) above) there is an $\alpha \in \bar{u}(L_2^*)$ with $C_D(\bar{u}(i), \alpha) = \varepsilon^s$. So $r | \mathrm{ord}(\alpha)$. But, as $\Gamma_{E \cdot L_2, u} / \Gamma_{E,u} \cong \mathcal{G}(E \cdot L_2/E) \cong \mathcal{G}(L_2/F)$ which has order r, $\mathrm{ord}(\alpha) = r$. Likewise, there is a $\beta \in \bar{u}(L_1^*)$ with $C_D(\bar{u}(j), \beta) = \varepsilon^r$, and $\mathrm{ord}(\beta) = s$. We have

$$C_D(\bar{u}(i), \alpha) = \varepsilon^s, \qquad C_D(\bar{u}(j), \alpha) = 1 \quad (\text{as } \rho \text{ stabilizes } L_2),$$
$$C_D(\bar{u}(i), \beta) = 1, \qquad C_D(\bar{u}(j), \beta) = \varepsilon^r,$$
$$C_D(\alpha, \beta) = 1, \qquad C_D(\bar{u}(i), \bar{u}(j)) = \varepsilon^t.$$

From these formulas it follows $\langle \alpha \rangle \cap \langle \beta \rangle = 1$, so $\{\alpha, \beta\}$ is a base of the abelian group H. Furthermore, if we write $r\bar{u}(i) = n_1\alpha + m_1\beta$ in H, we have $1 = C_D(\bar{u}(i), r\bar{u}(i)) = \varepsilon^{n_1 s}$, so $r | n_1$ and $n_1\alpha = 0$. Thus,

$$r\bar{u}(i) = m_1\beta \quad \text{and, likewise,} \quad s\bar{u}(j) = m_2\alpha,$$

for some integers m_1 and m_2. Hence, $G = G_1 \times G_2$, where $G_1 = \langle \beta, \bar{u}(i) \rangle$ and $G_2 = \langle \alpha, \bar{u}(j) \rangle$. Note that

$$\varepsilon^{rt} = C_D(r\bar{u}(i), \bar{u}(j)) = C_D(m_1\beta, \bar{u}(j)) = \varepsilon^{-m_1 r},$$
$$\varepsilon^{st} = C_D(\bar{u}(i), s\bar{u}(j)) = C_D(\bar{u}(i), m_2\alpha) = \varepsilon^{m_2 s};$$

so $m_1 \equiv -t \pmod{s}$ and $m_2 \equiv t \pmod{r}$. Thus, $\gcd(s, m_1) = \gcd(s, t)$ and $\gcd(r, m_2) = \gcd(r, t)$.

Consider the group $G_1 = \langle \beta, \bar{u}(i) \rangle$. Since the image of $\bar{u}(i)$ has order r in $G_1/\langle \beta \rangle$, $|G_1| = rs$ and

$$\mathrm{ord}(\bar{u}(i)) = r \, \mathrm{ord}(r\bar{u}(i)) = r \, \mathrm{ord}(m_1\beta) = rs/\gcd(s, m_1)$$
$$= rs/\gcd(s, t) = r \, \mathrm{lcm}(s, t)/t.$$

Therefore, G_1 has exponent

$$\mathrm{lcm}(\mathrm{ord}(\beta), \mathrm{ord}\,\bar{u}(i)) = \mathrm{lcm}(s, r \, \mathrm{lcm}(s, t)/t) = \mathrm{lcm}(st, rs, rt)/t = l.$$

Since the abelian group G_1 has two generators its rank is at most 2; thus, its invariant factors are determined by its order and exponent. That is, $G_1 \cong (\mathbf{Z}/l\mathbf{Z}) \times (\mathbf{Z}/d\mathbf{Z})$, where $d = |G_1|/\exp(G_1) = rs/l$. An analogous calculation shows that G_2 has the same order and exponent as G_1. Thus,

$$G = G_1 \times G_2 \cong G_1 \times G_1 \cong (\mathbf{Z}/l\mathbf{Z})^2 \times (\mathbf{Z}/d\mathbf{Z})^2,$$

as desired. $\quad \square$

An interesting feature of this lemma is the way the relative value group depends on $\bar{h}$ as well as r and s.

We now construct fields F and K so that the lemma can be applied in more than one way. Let k be any field containing a primitive rs-root of unity ε, so $\mathrm{char}(k) \nmid rs$. Let $F_0 = k(z_1, z_2)$ and $K_0 = k(z_1^{1/r}, z_2^{1/s})$, where z_1 and z_2 are algebraically independent over k. Let w be the valuation obtained by restricting to F_0 the standard valuation on $k((z_1))((z_2))$. Note that w has a totally ramified, hence unique, extension to K_0. We enlarge F_0 and K_0 to get more valuations and to obtain a convenient element of norm 1. Let

$$\delta = N_{K_0/F_0}(1 - z_1^{1/r} - z_2^{1/s}) = \prod_{m=1}^{r} \prod_{n=1}^{s} (1 - \varepsilon^{sm} z_1^{1/r} - \varepsilon^{rn} z_2^{1/s}) \in k[z_1, z_2].$$

Note that since δ factors in $k[z_1^{1/r}, z_2^{1/s}]$ into a product of rs distinct (and nonassociate) irreducibles, every irreducible factor of δ in $k[z_1, z_2]$ occurs with multiplicity 1. (In fact, one can check that δ is irreducible in $k[z_1, z_2]$.)

Let $\gamma = \delta^{1/rs}$, and let $F = F_0(\gamma), K = K_0(\gamma) = K_0 \cdot F$. It is clear from the prime factorization of δ that $[F : F_0] = rs$. Since δ is a 1-unit of (F_0, w), w has rs different extensions $w_1, \ldots, w_{rs}$ to F, and the images of γ in the residue fields $\overline{F}_{w_m}$ range over the powers of ε; number the w_m so that $\overline{\gamma} = \varepsilon^m$ in $\overline{F}_{w_m}$, $1 \leq m \leq rs$. Also, K_0 is linearly disjoint to F over F_0, as (K_0, w) is totally ramified over (F_0, w) while each (F, w_m) is an immediate extension of (F_0, w). Hence $[K : F] = rs$, K is Galois over F with $\mathcal{G}(K/F) \cong \mathcal{G}(K_0/F_0) \cong (\mathbf{Z}/r\mathbf{Z}) \times (\mathbf{Z}/s\mathbf{Z})$, and each w_m has a totally ramified (hence unique) extension to K. Let $\{\rho, \sigma\}$ be any base of $\mathcal{G}(K/F)$ with $\mathrm{ord}(\rho) = r$ and $\mathrm{ord}(\sigma) = s$. Let

$$h = \gamma/(1 - z_1^{1/r} - z_2^{1/s}) \in K.$$

Then, $N_{K/F}(h) = \gamma^{rs}/N_{K_0/F_0}(1 - z_1^{1/r} - z_2^{1/s}) = \delta/\delta = 1$.

PROPOSITION 5.8.   *Let D be the division algebra constructed as in (5.2) and (5.3) from the K, F, ρ, σ, and h just defined. Then, if $\gcd(r, s) > 1$, D is not a symbol algebra, nor is D isomorphic to a tensor product of symbol algebras.*

PROOF.   For each m with $1 \leq m \leq rs$, we can apply Lemma 5.6 with w_m for w, obtaining a valuation u_m on D which is totally ramified over $Z(D) = E$. Since $\bar{h} = \varepsilon^m/1 = \varepsilon^m$ in $\overline{K}_{w_m}$, we have $t = m$ in (5.4). Hence, the lemma shows $\Gamma_{D,u_1}/\Gamma_{E,u_1} \cong (\mathbf{Z}/rs\mathbf{Z})^2$, while for any prime p dividing both r and s, $\Gamma_{D,u_p}/\Gamma_{E,u_p} \cong (\mathbf{Z}/(rs/p)\mathbf{Z})^2 \times (\mathbf{Z}/p\mathbf{Z})^2$. If D were a symbol algebra or a tensor product of such algebras, then it would have an armature $\mathcal{D}$. Two applications of (3.3) show that $\mathcal{D} \cong (\mathbf{Z}/rs\mathbf{Z})^2$ and $\mathcal{D} \cong (\mathbf{Z}/(rs/p)\mathbf{Z})^2 \times (\mathbf{Z}/p\mathbf{Z})^2$, a contradiction. $\square$

REMARKS 5.9.   (i) For the D in (5.8), $[D : E] = (rs)^2$, and Proposition 4.7(i) with the valuation u_1 shows that $\mathrm{ord}[D] = rs$ in $\mathrm{Br}(F)$. Also, an application of $(4.7)(v)$ with the valuation u_p shows that D is not split by any cyclic Galois extension of E. As m ranges between 1 and rs, the relative value groups $\Gamma_{D,u_m}/\Gamma_{F,u_m}$ range over all the groups described in (5.6), for all values of t. These are precisely all the abelian groups which admit a nondegenerate symplectic pairing and have a Lagrangian isomorphic to $(\mathbf{Z}/r\mathbf{Z}) \times (\mathbf{Z}/s\mathbf{Z})$. (When $\gcd(r, s) = 1$, $(\mathbf{Z}/r\mathbf{Z})^2 \times (\mathbf{Z}/s\mathbf{Z})^2$ is the only such group, but when r and s have a common factor there is more than one such group—that is what was needed for (5.8).) The smallest dimensional division algebra to which (5.8) applies occurs when $r = s = 2$, so that $[D : F] = 16$.

(ii) The D in Proposition 5.8 is totally ramified over $Z(D) = E$ with respect to each of the valuations u_m, which are not Henselian on E. The proposition shows that the Henselian hypothesis in Draxl's decomposition Theorem 3.7 cannot be dropped.

(iii) Proposition 5.8 also shows that a division algebra D over a field with Henselian valuation need not be isomorphic to a product of symbol algebras even if $\overline{D}$ is a field. For this use the Henselian valuation v on E, for which $\overline{D}_v = K$. In addition, if r and s have the same prime factors, it can be shown that there are no totally ramified field extensions of (E, v) in D, and that no totally ramified algebraic field extension of (E, v) splits D.

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